Quantum Field Theory **Physics 721**

Final Exam Three hours 11 December 2009

Please write your name or initials on each piece of paper you turn in.

1. [35 points total] Suppose we have a neutral spin-0 particle "s" that can interact with itself by a ϕ^3 interaction and can also interact with electrons, and the interaction Lagrangian is

$$\mathcal{L}_{int} = -\frac{1}{3!}\, g\phi^3 - f\,\bar{\psi}\psi\phi\ ,$$

where ψ is the Dirac field for the electron, ϕ is the real scalar field for particle "s", and f and g are real constants. Let the mass of the scalar be M and neglect the mass of the electron.

Draw the Feynman diagram [5 points] for the process

$$e^+ + e^- \to s + s\,.$$

Assume $g \gg f$ so that diagrams of $\mathcal{O}(f^2)$ can be ignored compared to diagrams of $\mathcal{O}(fg)$. Also assume that the total center-of-mass energy is at least $2M$. Calculate [30 points] the center-of-mass differential cross section $d\sigma/d\Omega$.

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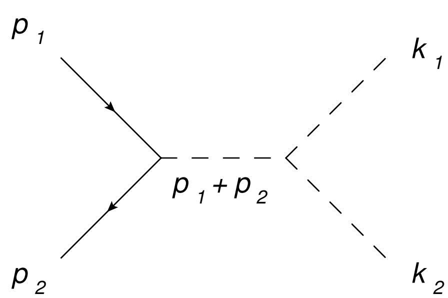
Feynman diagram (for one g and one f vertex)



Using $s = (p_1 + p_2)^2$,

$$i\mathcal{M} = (-ig)(-if)\bar{v}(p_2)u(p_1)\frac{i}{s - M^2}$$

Neglecting electron mass,

$$\sum_{spins} |\mathcal{M}|^2 = g^2 f^2 \frac{1}{(s - M^2)^2} \text{Tr}\ \not{p}_1 \not{p}_2 = g^2 f^2 \frac{2s}{(s - M^2)^2}$$

Cross section,

$$\sigma = \frac{1}{2(2E_1)(2E_2)} \int \frac{d^3k_1}{(2\pi)^3 2\omega_1} \frac{d^3k_2}{(2\pi)^3 2\omega_2} (2\pi)^4 \delta^4(P_{in} - P_{out}) \times \frac{1}{4} \sum_{spins} |\mathcal{M}|^2$$

Hence

$$\sigma = \frac{1}{2s} \int \frac{\omega_1^2 d\omega_1 d\Omega}{(2\pi)^2 4\omega_1\omega_2} \delta(E_{CM} - 2\omega_1) \times \frac{1}{4} \sum_{spins} |\mathcal{M}|^2$$

$$= \frac{1}{2s} \frac{1}{4\pi^2} \frac{1}{4} \frac{1}{2} \frac{1}{4} g^2 f^2 \frac{2s}{(s - M^2)^2}$$

Hence

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{g^2 f^2}{128\pi^2} \frac{1}{(s - M^2)^2}}$$

2. [35 points total] Suppose photons were massless scalar (spin-0) particles, and that the interaction Lagrangian density was

$$\mathcal{L}_{int} = -ig\,\bar{\psi}\gamma_5\psi\;\phi$$

where g is a real constant, ψ is the electron field, and ϕ is a real scalar field, describing the scalar particles which we will also call ϕ.

Find and simplify the matrix element for the analog of Compton scattering. (The analog of Compton scattering would be the process $\phi(k_1) + e^-(p_1) \to \phi(k_2) + e^-(p_2)$.)

Break down the problem as

a) Draw the (two) lowest order non-trivial Feynman diagrams for this process.

b) Write down the scattering amplitude $\mathcal{M}$ corresponding to the Feynman diagrams.

c) Simplify the scattering amplitude. I believe it can be written as something like

$$\mathcal{M} = (const.)\left(\frac{1}{\omega_1} \pm \frac{1}{\omega_2}\right)\bar{u}(p_2)\,\not{k}_1\,u(p_1) \equiv A \times \bar{u}(p_2)\,\not{k}_1\,u(p_1)\,,$$

where ω_1 and ω_2 are the energies of the incoming and outgoing ϕ's in the target rest frame (lab). Find A. (You are not asked, for this problem and for today, to continue after finding $\mathcal{M}$.)

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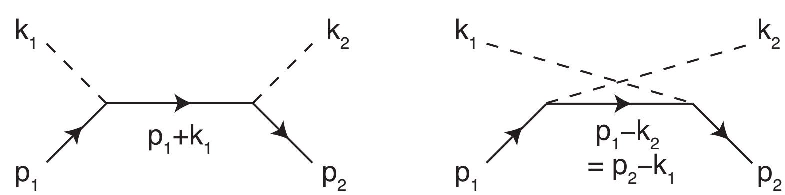
a)



b) and c)

$$\begin{aligned}
i\mathcal{M} &= i(g)^2\,\bar{u}(p_2)\left\{\gamma_5\frac{\not{p}_1 + \not{k}_1 + m}{(p_1+k_1)^2 - m^2}\gamma_5 + \gamma_5\frac{\not{p}_2 - \not{k}_1 + m}{(p_1-k_2)^2 - m^2}\gamma_5\right\}u(p_1) \\
&= ig^2\,\bar{u}(p_2)\left\{\frac{-\not{k}_1}{2m\omega_1} + \frac{\not{k}_1}{-2m\omega_2}\right\}u(p_1) \\
&= \frac{-ig^2}{2m}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\bar{u}(p_2)\,\not{k}_1\,u(p_1) \equiv iA\,\bar{u}(p_2)\,\not{k}_1\,u(p_1)
\end{aligned}$$

3. [10 points] From analyzing the interaction Lagrangian of question 1, what are the mass dimensions of f and g (in our units where $\hbar = 1$ and $c = 1$)?

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Lagrangians have units of energy, hence in mass dimensions the dimensions of the Lagrangian and Lagrangian density are

$$dim\{L\} = M \qquad \text{and} \qquad dim\{\mathcal{L}\} = M^4$$

From the known mass terms in the free Lagrangian ($m^2\phi^2$ or $m\bar{\psi}\psi$) one gets

$$dim\{\phi\} = M \qquad \text{and} \qquad dim\{\psi\} = M^{3/2} \ .$$

Thus

$$dim\{g\}\cdot dim\{\phi^3\} = M^4 \quad \text{and} \quad dim\{f\}\cdot dim\{\phi\}\cdot dim\{\psi^2\} = M^4$$

Thus

$$dim\{g\} = M^1 \qquad \text{and} \qquad dim\{f\} = M^0 \ .$$

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4. [10 points] For the interaction Lagrangian of question 2,

$$\mathcal{L}_{int} = -ig\,\bar{\psi}\gamma_5\psi\ \phi$$

where ψ is the electron field and ϕ is a real scalar field, show that g is real (or not) if $\mathcal{L}$ is hermitian.

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$$\mathcal{L}_{int} = -ig\,\psi^\dagger\gamma^0\gamma_5\psi\ \phi$$

Thus

$$\mathcal{L}_{int}^\dagger = +ig^*\,\psi^\dagger\gamma_5\gamma^0\psi\ \phi = -ig^*\,\psi^\dagger\gamma^0\gamma_5\psi\ \phi$$

where we used that γ^0, γ^5, and ϕ are hermitian. Thus since $\mathcal{L}_{int}$ is also hermitian,

$$g = g^* \ .$$

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5. [10 points] Given a two particle to two particle process where the incoming momenta are p_1 and p_2 and the outgoing momenta are p_3 and p_4, one has $p_1 + p_2 = p_3 + p_4$. Define

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2 \ .$$

Show that

$$s + t + u = 0$$

for the case that all four particles are massless.

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For massless particles,

$$s = 2p_1\cdot p_2, \qquad t = -2p_1\cdot p_3, \qquad u = -2p_1\cdot p_4 \ ,$$

$$s + t + u = 2p_1\cdot(p_2 - p_3 - p_4) = -2p_1^2 = 0 \ .$$

where we used momentum conservation, $p_2 - p_3 - p_4 = -p_1$.

By the way, for massive particles it is almost as easy.

$$s = m_1^2 + m_2^2 + 2p_1\cdot p_2, \qquad t = m_1^2 + m_3^2 - 2p_1\cdot p_3, \qquad u = m_1^2 + m_4^2 - 2p_1\cdot p_4 \ ,$$

$$\begin{aligned} s + t + u &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1\cdot(p_2 - p_3 - p_4) \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2p_1^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \end{aligned}$$