Midterm exam, Classical E&M II, Physics 611, Fall 2015

1. [35 points total] A beam of light, traveling in the z-direction, passes through four filters, labeled 0 through 3. ``Filter'' 0 passes all the light; filter 1 passes only light plane polarized in the x direction; filter 2 passes only light plane polarized at 45° between the x and y axes; and filter 3 passes only right-handed polarized light.

You measure that filter 1 passes 80% as much intensity as filter 0; filter 2 passes 84.6% as much intensity as filter 0; and filter 3 passes 30.0% as much intensity as filter 0. [corrected]

a) Find the polarization of the beam. Specifically, writing the electric field of the initial beam as

$$\vec{E} = (a_1\, \hat{x} + a_2\, e^{i\phi}\, \hat{y})\, e^{i(kz-\omega t)},$$

with a_1 and a_2 real and positive, and ϕ real, find a_2/a_1 and ϕ.

b) The full intensity of the beam is 45 Watts/m^2. Find a_1 in appropriate units.

From class notes or elsewhere (relations equivalent to Stokes parameter relations):
I1 / I0=a1^2/(a1^2+a2^2)
a2^2/a1^2=I1 / I0 - 1
cosphi = (I2 / I0 - 1/2) (a2/a1 + a1/a2)
sinphi = (I3 / I0 - 1/2) (a2/a1 + a1/a2)

Intensity = $\epsilon_0\, c \,\langle \vec{E}^2 \rangle = 0.5\, \epsilon_0\, c\,(a_1^2+a_2^2) = 0.5\, \frac{5}{4}\, \epsilon_0\, c\, a_1^2 = 45$ W.

```
I0overI1 = 1.00 / 0.80;
I2overI0 = 0.846;
I3overI0 = 0.300;

a2overa1 = Sqrt[I0overI1 - 1]; Print["a) a₂/a₁ = ", a2overa1]
cosphi = (I2overI0 - 1 / 2) ( a2overa1 + 1 / a2overa1) ;
magphi = ArcCos[cosphi] * 180 / π ;
sinphi = (I3overI0 - 1 / 2) ( a2overa1 + 1 / a2overa1) ;
phi = ArcSin[sinphi] * 180 / π ; Print["    ϕ = ", phi, "°"]

Intensity = 45;
c = 3 * 10^8;
epsilon = 8.854 * 10^-12;
a1 = Sqrt[(8 / 5) Intensity / (c epsilon)];
(* Check units: J/(s m²) s/m N m²/C² = N²/C². Good *)
Print["b) a₁ = ", a1, " N/C"]
```

a) $a_2/a_1 = 0.5$

$\phi = -30.°$

b) $a_1 = 164.64$ N/C

2. [30 points total] Another beam of light, also traveling in the z-direction with

$$\vec{E} = (a_1\, \hat{x} + a_2\, e^{i\phi}\, \hat{y})\, e^{i(kz-\omega t)},$$

has equal x and y amplitudes $a_1 = a_2 = a$.

a) The polarization vector will, as time passes, trace out an ellipse in the x-y plane (take z=0) with its major axis at an angle θ to the x axis. What is θ?

b) Sketch the polarization ellipse for the cases ϕ = 60° and ϕ = 30°. (You can put them on the same sketch; label which is which.)

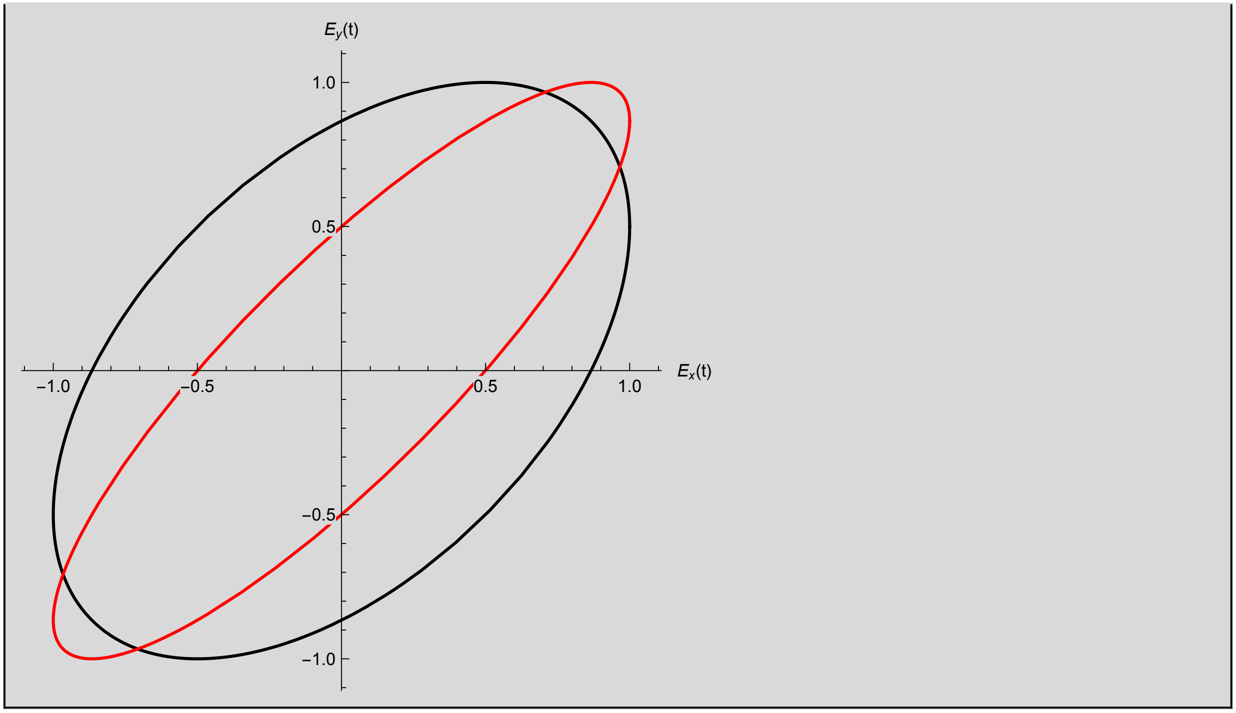
θ can be worked out by trying examples with a *Mathematica* code, or by finding the time t that maximizes $(E_x^2+E_y^2)$ and then finding the angle $\theta = \text{ArcTan}[E_y/E_x]$ at this time.

$E_x=\text{Cos}(\omega t)$
$E_y=\text{Cos}(\omega t-\phi)$
$(E_x^2+E_y^2) = \text{Cos}^2(\omega t)+\text{Cos}^2(\omega t-\phi)$
derivative= $-\text{Sin}(2\omega t) - \text{Sin}(2(\omega t-\phi))$
Hence $(\omega t-\phi) = -\omega t$ or $(\omega t-\phi) = -\omega t + 90°$. The latter corresponds (you will find) to the minimum, so take the former.
Hence $\omega t = \phi/2$, and at this time E_x and E_y are the same.
$\theta = \text{ArcTan}[E_y/E_x] = 45°$.

```
Print["a)  θ = 45°"]
ex[t_] := Cos[2 Pi t]
ey[t_] := Cos[ 2 Pi t - Pi/3]
plota = ParametricPlot[{ex[t], ey[t]},
   {t, 0, 1}, AxesLabel → {"Ex(t)", "Ey(t)"}, PlotStyle → Black];
ex[t_] := Cos[2 Pi t]
ey[t_] := Cos[ 2 Pi t - Pi/6]
plotb = ParametricPlot[{ex[t], ey[t]},
   {t, 0, 1}, AxesLabel → {"Ex(t)", "Ey(t)"}, PlotStyle → Red];
Print["b) The 30° case is the narrower ellipse."]
Show[plota, plotb]
```

a) $\theta = 45°$

b) The 30° case is the narrower ellipse.



3. [35 points total] A particle of charge q travels in a straight line at speed $v \neq 0$, and passes at distance a at closest approach to a certain point. Work out the (retarded) electromagnetic potentials in Lorenz gauge at this point at the time of closest approach.

For definiteness, say that the charged particle is moving along the z axis and passes through the origin at time zero, and that the point in question is on the x axis, distance a from the origin.

a) Show that only one point along the charged particle's path contributes to the desired potentials at time $t = 0$. Do so by finding that point along the path.

b) Find the scalar potential at the point in question, $\Phi = \Phi$ (x=a y=0, z=0, t=0), in terms of q, a, v, and standard electromagnetic constants.

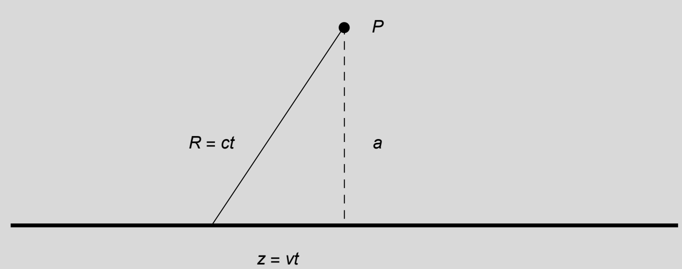
c) Find the vector potential at the same point.

The potential at point P at time 0 is emitted at time t (must be negative), and from the figure below it is the point where $|t| = |z| / v = R / c =$ Sqrt[$z^2 + a^2$]/c .
Solve to get $z = - \beta \gamma a$, for $\beta = v/c$ and $\gamma = 1/$Sqrt[$1 - v^2/c^2$], for the point of emission along the parti-

cle’s path. BTW, for this point, $R = \gamma a$.

In Lorenz gauge, $\Phi = 1/4\pi\epsilon_0 \int d^3x'\, \rho(x', t' = t - R/c)/|x' - x|$ and $\rho = q\,\delta^3(\vec{x}' - v\hat{z}t')$.
Get $\Phi = q / (4\pi\epsilon_0 \gamma a)$.
A similar consideration works for $\vec{A}$.

```
Show[Graphics[{Black, Thick, Line[{{-2, 0}, {2, 0}}]}],
 Graphics[Text["a", {0.20, 0.5}]],
 Graphics[Text["P", {0.20, 1.2}]],
 Graphics[Text["●", {0.0, 1.2}]],
 Graphics[{Black, Black, Line[{{-0.8, 0}, {0, 1.2}}]}],
 Graphics[{Black, Dashed, Line[{{0, 1.2}, {0, 0}}]}],
 Graphics[Text["z = vt", {-0.4, -0.2}]], Graphics[Text["R = ct", {-0.8, 0.5}]]
]
Print["a)  z = - β γ a."]
Print["b)  Φ = 1/(4 π ϵ0) q/(γ a)."]
Print["c)  A⃗ = μ0/(4 π) (q v ẑ)/(γ a)."]
```



a) $z = -\beta\gamma a$.

b) $\Phi = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\gamma a}$.

c) $\vec{A} = \dfrac{\mu_0}{4\pi}\dfrac{q v \hat{z}}{\gamma a}$.