Quantum Field Theory **Physics 721**

Midterm Exam **80 minute time limit** **15 October 2009**

☞ Please write your name or initials on each piece of paper you turn in.

1. [40 points] a) For the current $\bar{\psi}(x)\gamma^\mu\psi(x)$, where $\psi(x)$ is a Dirac field, evaluate

$$\langle p_2, s_2| \, : \bar{\psi}(x)\gamma^\mu\psi(x) : \, |p_1, s_1\rangle \, ,$$

when both the initial and final state are defined using $|p_i, s_i\rangle = a^\dagger(p_i, s_i)|0\rangle$. (The colons indicate normal ordering.)
b) Do the same for the case that both the initial and final state are defined using $|p_i, s_i\rangle = b^\dagger(p_i, s_i)|0\rangle$.
c) Show, using the definition $v(p,s) = i\gamma^2 u^*(p,s) = i\gamma^2\gamma^0\bar{u}^T(p,s)$, that

$$\bar{v}(p_1, s_1)\gamma^\mu v(p_2, s_2) = \bar{u}(p_2, s_2)\gamma^\mu u(p_1, s_1) \, .$$

. . . .

a)

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} \left(a_{ps} u_{ps} e^{-ipx} + b^\dagger_{ps} v_{ps} e^{ipx} \right) \tag{1}$$

With the normal ordering, the $b^\dagger$ will go to the left, anticommute with the $a_{p_2 s_2}$ that is already on the left, and annihilate the left vacuum. Hence only the a_{ps} gives something, leaving only the spinor wave function and the exponential:

$$\sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} a_{ps} u_{ps} e^{-ipx} \, a^\dagger_{p_1 s_1}|0\rangle = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} \, (2\pi)^3 2E_p \delta_{ss_1} \delta^3(\vec{p} - \vec{p}_1) \, u_{ps} e^{-ipx}|0\rangle$$
$$= u_{p_1 s_1} \, e^{-ip_1 x} \, |0\rangle$$

For $\bar{\psi}(x)$ in the present case only the $a^\dagger$ will contribute, the manipulation will be similar, and the overall matrix element will be

$$_a\langle p_2, s_2| \, : \bar{\psi}(x)\gamma^\mu\psi(x) : \, |p_1, s_1\rangle_a = \bar{u}_{p_2 s_2}\gamma^\mu u_{p_1 s_1} \, e^{i(p_2 - p_1)x} \, . \tag{2}$$

b) For the antiparticles, the b from $\bar{\psi}(x)$ and the $b^\dagger$ from $\psi(x)$ will give the result. The crucial difference from the previous case is the "$-$" sign that comes from the normal ordering,

$$: b_{p's'} b^\dagger_{ps} : = -b^\dagger_{ps} b_{p's'}, \tag{3}$$

so that

$$_b\langle p_2, s_2| \, : \bar{\psi}(x)\gamma^\mu\psi(x) : \, |p_1, s_1\rangle_b = -\bar{v}_{p_1 s_1}\gamma^\mu v_{p_2 s_2} \, e^{-i(p_2 - p_1)x} \, . \tag{4}$$

c) Also using $\bar{v} = u^T(i\gamma^2\gamma^0)$ and $(i\gamma^2\gamma^0)\gamma^\mu(i\gamma^2\gamma^0) = (\gamma^\mu)^T$ (which you should prove if you are uncertain),

$$\bar{v}(p_1, s_1)\gamma^\mu v(p_2, s_2) = u^T(p_1, s_1)(i\gamma^2\gamma^0)\gamma^\mu(i\gamma^2\gamma^0)\bar{u}^T(p_2, s_2)$$
$$= u^T(p_1, s_1) \, (\gamma^\mu)^T \, \bar{u}^T(p_2, s_2) = \bar{u}(p_2, s_2)\gamma^\mu u(p_1, s_1) \, . \tag{5}$$

2. [35 points] Consider the Lagrangian density

$$\mathcal{L} = \bar{\psi}\,(i\,\partial\!\!\!/ - M)\,\psi + \frac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - g\,\bar{\psi}\gamma_\mu\gamma_5\psi\,\partial^\mu\phi\,,$$

where ψ is a Dirac field, ϕ is a Hermitian scalar field, and m, M, and g are real constants.
a) Consider the transformation $\psi(x) \to e^{-i\alpha}\psi(x)$, $\bar{\psi}(x) \to e^{i\alpha}\bar{\psi}(x)$, and $\phi(x) \to \phi(x)$, where α is a real constant. Is the Lagrangian invariant under this transformation? Is the action invariant under this transformation? If yes to either question, find the conserved current that goes along with this transformation.
b) Consider the transformation $\psi(x) \to \psi(x+\beta)$, $\bar{\psi}(x) \to \bar{\psi}(x+\beta)$, and $\phi(x) \to \phi(x+\beta)$, where β is a real constant 4-vector. Is the Lagrangian invariant under this transformation? Is the action invariant under this transformation? If yes to either question, find the conserved current that goes along with this transformation.
c) Consider the transformation $\psi(x) \to \psi(x) + \chi$, $\bar{\psi}(x) \to \bar{\psi}(x) + \bar{\chi}$, and $\phi(x) \to \phi(x)$, where χ is some constant 4-component column matrix. Is the Lagrangian invariant under this transformation? Is the action invariant under this transformation? If yes to either question, find the conserved current that goes along with this transformation.

. . . .

a) The Lagrangian and the action are both invariant under this transformation.
The changes in the fields for infinitesimal parameter α are

$$\Delta\psi = -i\alpha\psi\,, \qquad \Delta\bar{\psi} = +i\alpha\bar{\psi} \tag{6}$$

conserved current is

$$j_\mu \cdot \alpha = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\Delta\psi + (\psi \to \bar{\psi}) \tag{7}$$

$$j_\mu = \bar{\psi}\gamma_\mu\psi \tag{8}$$

b) The Lagrangian is not invariant because the argument has changed, but the action is. With the d^4x in the definition of the action, one can do a shift of variables and show the action is the same before and after the transformation. There are four parameters b^ν that can be taken infinitesimal, and the changes in the fields are

$$\Delta\psi = b^\nu\partial_\nu\psi, \quad \Delta\bar{\psi} = b^\nu\partial_\nu\bar{\psi}, \quad \Delta\phi = b^\nu\partial_\nu\phi \tag{9}$$

and the Lagrangian changes by $\Delta\mathcal{L} = b^\nu\partial_\nu\mathcal{L}$.
The current has the Lorentz index that a conserved current should have, but now also has a second index because the is one current for each b^ν,

$$J_{\mu\nu} \cdot b^\nu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\Delta\psi + (\psi \to \bar{\psi}) + (\psi \to \phi) - g_{\mu\nu}\mathcal{L}\,b^\nu \tag{10}$$

$$J_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\partial_\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$$

$$= \bar{\psi}\gamma_\mu\partial_\nu\psi + \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{11}$$

(Commonly the notation for this current is $T_{\mu\nu}$.)
c) Neither the Lagrangian nor the action is invariant.

3. [25 points] Given the Lagrangian for a complex Klein-Gordon field,

$$\mathcal{L} = a\left[(\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi\right]$$

(Note: should be $a = 1$, but can work with general a also.)
a) Find the canonical momenta $\pi(x)$, corresponding to $\phi(x)$, and $\pi^*(x)$, defined to be the canonical momentum corresponding to $\phi^*(x)$.
b) Find the Hamiltonian density in terms of the correct (for a Hamiltonian) fields and momenta.

. . . .

a)

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = a\dot{\phi}^*(x) \tag{12}$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*(x)} = a\dot{\phi}(x) \tag{13}$$

b)

$$\mathcal{H} = \pi(x)\dot{\phi}(x) + \pi^*(x)\dot{\phi}^*(x) - \mathcal{L} \tag{14}$$

$$= a\left[|\dot{\phi}(x)|^2 + |\vec{\nabla}\phi(x)|^2 + m^2|\phi(x)|^2\right] \tag{15}$$