

### Homework 3

Physics 721-QFT  
handed out 29 Sep. 2009

1. For Dirac fields define a spin operator

$$\vec{S} = \int d^3x \psi^\dagger(x) \frac{1}{2} \vec{\sigma} \psi(x)$$

a) Show that the  $S_i$  commute the way spin operators should, *i.e.*, show that  $[S_1, S_2] = iS_3$ , *etc.*

b) For single particle states  $|\vec{p}, s\rangle_a = a^\dagger(\vec{p}, s) |0\rangle$  moving in the z-direction, show that

$$S_z |\vec{p}, s\rangle_a = s |\vec{p}, s\rangle_a$$

where we label the spin using  $s = \pm \frac{1}{2}$ .

c) Find the corresponding result for the states  $|\vec{p}, s\rangle_b = b^\dagger(\vec{p}, s) |0\rangle$ .

Optional: would you like to relabel the states, or would you prefer to just remember the result?

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a) There are Dirac indices. Keep track of them mentally.

$$[S_i, S_j] = \int d^3x d^3x' \left[ \psi^\dagger(x) \frac{1}{2} \sigma_i \psi(x), \psi^\dagger(x') \frac{1}{2} \sigma_j \psi(x') \right] \quad (1)$$

Use theorems:

$$[AB, CD] = A[B, CD] + [A, CD]B \quad ; \quad [B, CD] = -C\{B, D\} + \{B, C\}D \quad (2)$$

$$\begin{aligned} [S_i, S_j] &= \int d^3x d^3x' \left( \psi^\dagger(x) \frac{1}{2} \sigma_i \left[ \psi(x), \psi^\dagger(x') \frac{1}{2} \sigma_j \psi(x') \right] + \left[ \psi^\dagger(x), \psi^\dagger(x') \frac{1}{2} \sigma_j \psi(x') \right] \frac{1}{2} \sigma_i \psi(x) \right) \\ &= \int d^3x d^3x' \left( \psi^\dagger(x) \frac{1}{2} \sigma_i \left\{ \psi(x), \psi^\dagger(x') \right\} \frac{1}{2} \sigma_j \psi(x') - \psi^\dagger(x') \frac{1}{2} \sigma_j \left\{ \psi^\dagger(x), \psi(x') \right\} \frac{1}{2} \sigma_i \psi(x) \right) \\ &= \int d^3x \psi^\dagger(x) \left[ \frac{1}{2} \sigma_i, \frac{1}{2} \sigma_j \right] \psi(x) = i\epsilon_{ijk} \int d^3x \psi^\dagger(x) \frac{1}{2} \sigma_k \psi(x) = i\epsilon_{ijk} S_k \end{aligned} \quad (3)$$

b) Use (with a similar equation for  $\psi^\dagger$ ),

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} \left( a_{ps} u_{ps} e^{-ipx} + b_{ps}^\dagger v_{ps} e^{ipx} \right), \quad (4)$$

and with normal ordering, get

$$S_z = \sum_{s, s'} \int \frac{d^3p}{(2\pi)^3 (2E_p)^2} \left( a_{ps'}^\dagger a_{ps} u_{ps'}^\dagger \frac{1}{2} \sigma_z u_{ps} - b_{ps}^\dagger b_{ps'} v_{ps'}^\dagger \frac{1}{2} \sigma_z v_{ps} \right) \quad (5)$$

Some terms were dropped using  $v_{-p, s'}^\dagger \frac{1}{2} \sigma_k u_{ps} = 0$  (for states with  $\vec{p} \parallel \hat{z}$ ).

Show that for states with  $\vec{p} \parallel \hat{z}$ ,

$$u_{ps'}^\dagger \frac{1}{2} \sigma_z u_{ps} = 2E_p s \delta_{ss'} \quad ; \quad v_{ps'}^\dagger \frac{1}{2} \sigma_z v_{ps} = -2E_p s \delta_{ss'} \quad (6)$$

Hence

$$S_z = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_p} s \left( a_{ps}^\dagger a_{ps} + b_{ps}^\dagger b_{ps} \right) \quad (7)$$

Hence

$$S_z |\vec{p}, s\rangle_a = s |\vec{p}, s\rangle_a \quad (8)$$

c) Hence also

$$S_z |\vec{p}, s\rangle_b = s |\vec{p}, s\rangle_b \quad (9)$$

2. Include electromagnetic interactions in the Dirac equation in analogy to the classical result that the Hamiltonian with interactions is obtained from the one without interactions by  $p^\mu \rightarrow p^\mu - eA^\mu$ , where  $e$  is the charge of the particle. Quantum mechanically, one can also write this as  $i\partial^\mu \rightarrow i\partial^\mu - eA^\mu$ . Hence a Dirac wave function satisfies

$$(i\partial - eA - m)\psi(x) = 0.$$

Show that the conjugate solution  $\psi^C = i\gamma^2\psi^*(x)$  satisfies the same equation, but with the opposite charge.

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$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi(x) = 0 \tag{10}$$

Complex conjugate and multiply from the left by  $i\gamma^2$ ,

$$i\gamma^2(-i\gamma^{\mu*}\partial_\mu - e\gamma^{\mu*}A_\mu - m)\psi^*(x) = 0 \tag{11}$$

Recall or work out the identity  $(i\gamma^2)\gamma^{\mu*} = -\gamma^\mu(i\gamma^2)$ ,

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)(i\gamma^2)\psi^*(x) = 0 \tag{12}$$

or

$$(i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi^C(x) = 0 \tag{13}$$

3. [Peskin and Schroeder 3.2] Prove the *Gordon identity*,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p') \left[ \frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \right] u(p)$$

where  $q \equiv p' - p$ . If you think too many people have already proved the Gordon identity, you may instead find and prove the analogous identity involving  $\bar{u}(p')\gamma^\mu v(p)$ .

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**Start with**

$$\bar{u}(p') i\sigma^{\mu\nu} q_\nu u(p) = -\frac{1}{2}\bar{u}(p') (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'_\nu - p_\nu) u(p) \quad (14)$$

$$= -\frac{1}{2}\bar{u}(p') (\gamma^\mu \not{p}' - \gamma^\mu \not{p} - \not{p}'\gamma^\mu + \not{p}\gamma^\mu) u(p) \quad (15)$$

$$= -\frac{1}{2}\bar{u}(p') (2p'^\mu - 2\gamma^\mu \not{p} - 2\not{p}'\gamma^\mu + 2p^\mu) u(p) \quad (16)$$

$$= +2m \bar{u}(p') \gamma^\mu u(p) - \bar{u}(p') (p+p')^\mu u(p) \quad (17)$$

Reorganization gives the stated Gordon identity.

For the identity with the antiparticle spinor, start with

$$\bar{u}(p') i\sigma^{\mu\nu} (p' + p)_\nu v(p) = -\frac{1}{2}\bar{u}(p') (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p'_\nu + p_\nu) v(p) \quad (18)$$

$$= -\frac{1}{2}\bar{u}(p') (\gamma^\mu \not{p}' + \gamma^\mu \not{p} - \not{p}'\gamma^\mu - \not{p}\gamma^\mu) v(p) \quad (19)$$

$$= -\frac{1}{2}\bar{u}(p') (2p'^\mu + 2\gamma^\mu \not{p} - 2\not{p}'\gamma^\mu - 2p^\mu) v(p) \quad (20)$$

$$= +2m \bar{u}(p') \gamma^\mu v(p) - \bar{u}(p') (p' - p)^\mu v(p) \quad (21)$$

Organize into

$$\bar{u}(p')\gamma^\mu v(p) = \bar{u}(p') \left[ \frac{(p' - p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p' + p)_\nu}{2m} \right] v(p) \quad (22)$$

4. [Cf. Peskin & Schroeder 3.4] The Dirac analog of a real Klein-Gordon field is the *Majorana field*, which is a Dirac field that satisfies the requirement

$$\psi = \psi^C = i\gamma^2\psi^* .$$

- a) Show that with this requirement, there is only one type of creation or annihilation operator. Or, in the notation of ordinary Dirac fields, show that  $a_{ps} = b_{ps}$ . Use the definition  $v_s(p) \equiv i\gamma^2 u_s^*(p)$ .
- b) Show that the ordinary mass term,  $-m\bar{\psi}\psi$ , is identically zero (*i.e.*, zero without using any equations of motion) for Majorana fields, *if the fields commute*.
- c) Show that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

leads to the standard Dirac equation for the Majorana field.

- d) (Optional) Find the Hamiltonian density for the free Majorana field, and express the Hamiltonian  $H = \int d^3x \mathcal{H}$  using creation and annihilation operators. You may need to think about ordering problems in the definitions of the canonical momentum density and Hamiltonian density.

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4a)

$$\psi(x) = \sum_s \int (d^3 p) \left( a_{ps} u_{ps} e^{-ipx} + b_{ps}^\dagger v_{ps} e^{ipx} \right) \quad (23)$$

$$\psi^C(x) = i\gamma^2 \psi^*(x) = \sum_s \int (d^3 p) \left( a_{ps}^\dagger v_{ps} e^{ipx} + b_{ps} u_{ps} e^{-ipx} \right) \quad (24)$$

Note that  $\psi^* = \psi^{\dagger T}$ ; that is how we get the daggers on the  $a$ 's and  $b$ 's. Also recall  $v_{ps} = i\gamma^2 u_{ps}^*$  and  $u_{ps} = i\gamma^2 v_{ps}^*$ .

From  $\psi = \psi^C$ , get

$$a_{ps} = b_{ps} \quad (25)$$

b) Work out that, in general,  $\bar{\psi}^C = \psi^T i\gamma^2 \gamma^0$ . In the present case  $\bar{\psi} = \bar{\psi}^C$ , and

$$\mathcal{L}_m = -m\bar{\psi}\psi = -m\psi^T i\gamma^2 \gamma^0 \psi = -m\psi_a \{i\gamma^2 \gamma^0\}_{ab} \psi_b \quad (26)$$

Thus,

$$C = i\gamma^2 \gamma^0 = i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \quad (27)$$

which is (clearly) antisymmetric. Then if  $\psi_a \psi_b$  is symmetric—which it is if the components commute—and  $C_{ab}$  is antisymmetric,

$$\mathcal{L}_m \equiv 0 \quad (28)$$

c)

$$\mathcal{L} = \frac{1}{2} \psi_a C_{ab} (i\gamma^\mu \partial_\mu - m)_{bc} \psi_c \quad (29)$$

$$\frac{\partial \mathcal{L}}{\partial \psi_d} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_d)} = 0 \implies \frac{1}{2} C_{ab} (i\gamma^\mu \partial_\mu - m)_{bc} \psi_c + \frac{1}{2} \psi_a C_{ab} m \mathbb{1}_{bd} + \frac{1}{2} (\partial_\mu \psi_a) C_{ab} (i\gamma^\mu)_{bd} = 0 \quad (30)$$

or

$$\frac{1}{2} C (i\gamma^\mu \partial_\mu - m) \psi + \frac{1}{2} m C^T \psi + \frac{1}{2} i\gamma^\mu{}^T C^T \partial_\mu \psi = 0 \quad (31)$$

Working out that  $\gamma^\mu{}^T C^T = C \gamma^\mu$  and remembering that  $C^T = -C$ , get

$$C (i\gamma^\mu \partial_\mu - m) \psi = 0 \quad (32)$$

which since  $C$  is invertible is the standard Dirac equation.

d)

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = -\frac{i}{2} \left( \psi^T C \gamma^0 \right)_a \quad (33)$$

$$\mathcal{H} = \dot{\psi}_a \pi_a - \mathcal{L} = +\frac{i}{2} \left( \psi^T C \gamma^0 \right) \dot{\psi} - \frac{1}{2} \psi^T C (i\gamma^\mu \partial_\mu - m) \psi = \dots = \frac{1}{2} \psi^\dagger (-i\vec{\alpha} \cdot \vec{\nabla} + m) \psi \quad (34)$$

$$\psi(x) = \sum_s \int (d^3 p) \left( a_{ps} u_{ps} e^{-ipx} + a_{ps}^\dagger v_{ps} e^{ipx} \right) \quad (35)$$

$$H = \int d^3 x \mathcal{H} \quad (36)$$

The rest of the calculation is much like that for the regular Dirac Hamiltonian, and one gets

$$H = \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{E_p}{2} \left( a_{ps}^\dagger a_{ps} - a_{ps} a_{ps}^\dagger \right) \equiv \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{E_p}{2} a_{ps}^\dagger a_{ps} \quad (37)$$