Homework 3 **Physics 721–QFT**

handed out 22 Sep. 2009

1. (a) For a complex scalar field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(a_{\vec{p}}\, e^{-ipx} + b^\dagger_{\vec{p}}\, e^{ipx} \right)$$

find the charge operator that goes along with the current

$$j^\mu = : i\phi^\dagger \overleftrightarrow{\partial}^\mu \phi : = : i(\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi) :$$

in terms of the creation and annihilation operators ($a^\dagger_{\vec{p}}$, $b^\dagger_{\vec{p}}$ and $a_{\vec{p}}, b_{\vec{p}}$, respectively).
(b) Show that the two categories of single particle states, $|\vec{p}\,\rangle_a = a^\dagger_{\vec{p}}\,|0\rangle$ and $|\vec{p}\,\rangle_b = b^\dagger_{\vec{p}}\,|0\rangle$, are both eigenstates of the charge operator, with opposite eigenvalues.

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a) Can choose $t = 0$

$$Q = \int d^3x\, j^0(x) =: i \left(\int d^3x \phi^\dagger(x)\dot{\phi}(x) - h.c. \right) : \tag{1}$$

$$= i : \int (d^3p) \int (d^3p') \int d^3x \left(a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} + b_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \right) (-i\omega_p) \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - b^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right) - h.c. : \tag{2}$$

$$= \frac{1}{2} \left(\int (d^3p) \left[a^\dagger_{\vec{p}} a_{\vec{p}} - a^\dagger_{-\vec{p}} b^\dagger_{\vec{p}} + b_{-\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}} \right] + h.c. \right) \tag{3}$$

$$\therefore \quad Q = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}} \right) \tag{4}$$

b)

$$Q|\vec{p}\,\rangle_a = \int \frac{d^3p'}{(2\pi)^3 2\omega_{p'}}\, a^\dagger_{\vec{p}'} a_{\vec{p}'}\, a^\dagger_{\vec{p}}\,|0\rangle = \int \frac{d^3p'}{(2\pi)^3 2\omega_{p'}}\, a^\dagger_{\vec{p}'} \left[(2\pi)^3 2\omega_p \delta^3(p'-p) + a^\dagger_{\vec{p}} a_{\vec{p}'} \right] |0\rangle = a^\dagger_{\vec{p}}|0\rangle \tag{5}$$

$$\therefore Q|\vec{p}\,\rangle_a = |\vec{p}\,\rangle_a \tag{6}$$

$Q|\vec{p}\,\rangle_b$ is the same except for the minus sign.

2. Show that the rapidity η, defined from the Lorentz parameter γ by $\cosh\eta = \gamma$, is additive for any two successive Lorentz transformations in the same direction.

(*I.e.*, If we label a Lorentz transformation by the value of the rapidity, show that $\Lambda(\eta_2) \circ \Lambda(\eta_1) = \Lambda(\eta_1 + \eta_2)$.)

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Choose axes so that the x-axis lines up with the direction of the Lorentz transformations. Then

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

and

$$\Lambda(\eta_2)\Lambda(\eta_1) = \begin{pmatrix} \cosh\eta_2 & \sinh\eta_2 & 0 & 0 \\ \sinh\eta_2 & \cosh\eta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 & 0 & 0 \\ \sinh\eta_1 & \cosh\eta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} \cosh\eta_2\cosh\eta_1 + \sinh\eta_2\sinh\eta_1 & \cosh\eta_2\sinh\eta_1 + \sinh\eta_2\cosh\eta_1 & 0 & 0 \\ \cosh\eta_2\sinh\eta_1 + \sinh\eta_2\cosh\eta_1 & \cosh\eta_2\cosh\eta_1 + \sinh\eta_2\sinh\eta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} \cosh(\eta_1+\eta_2) & \sinh(\eta_1+\eta_2) & 0 & 0 \\ \sinh(\eta_1+\eta_2) & \cosh(\eta_1+\eta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

$$= \Lambda(\eta_1 + \eta_2) \tag{11}$$