Homework 1

Physics 721–QFT

handed out 01 Sep. 2009

1. **(Review problem)** (a) Show that the following classical non-relativistic Lagrangian, for particles of mass m and charge e interacting with given external electromagnetic potentials $V(x)$ and $\vec{A}(x)$,

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - eV(x) + e\dot{\vec{x}} \cdot \vec{A}(x) \,,$$

leads to the expected Lorentz force law,

$$m\ddot{\vec{x}} = e\left(\vec{E} + \dot{\vec{x}} \times \vec{B}\right).$$

(A dot above a letter means a total derivative w.r.t. t.)
(b) While you are at it, show that the canonical momentum of the particle and the Hamiltonian are given by

$$\vec{p} = m\dot{\vec{x}} + e\vec{A}$$

and

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}(x)\right)^2 + eV(x) \,.$$

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(a)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \tag{1}$$

$$\frac{d}{dt}(m\dot{x}^i + eA^i) + e\frac{\partial V}{\partial x^i} - e\dot{x}^j \frac{\partial A^j}{\partial x^i} = 0 \tag{2}$$

$$m\ddot{x}^i + e\left(\frac{\partial V}{\partial x^i} + \frac{\partial A^i}{\partial t}\right) - e\left(\dot{x}^j\frac{\partial A^j}{\partial x^i} - \dot{x}^j\frac{\partial A^i}{\partial x^j}\right) = 0 \tag{3}$$

Know

$$E^i = -\frac{\partial V}{\partial x^i} - \frac{\partial A^i}{\partial t} \tag{4}$$

Recognize

$$\left(\dot{\vec{x}} \times \vec{B}\right)^i = \left(\dot{\vec{x}} \times \left(\vec{\nabla} \times \vec{A}\right)\right)^i = \dot{x}^j\frac{\partial A^j}{\partial x^i} - \dot{x}^j\frac{\partial A^i}{\partial x^j} \tag{5}$$

Thus

$$m\ddot{x}^i = e\left(E^i + \left(\dot{\vec{x}} \times \vec{B}\right)^i\right) \tag{6}$$

(b)

$$p^i \overset{def}{=} \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + eA^i \tag{7}$$

$$\begin{aligned} H = \vec{p}\cdot\dot{\vec{x}} - L &= \vec{p}\cdot\frac{1}{m}\left(\vec{p} - e\vec{A}\right) - \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + eV - \frac{e}{m}\left(\vec{p} - e\vec{A}\right)\cdot\vec{A} & (8) \\ &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + eV & (9) \end{aligned}$$

2. From the energy-momentum tensor for the scalar field, find the momentum operator $\vec{P}$. Then show that if the wave function for a single particle state $|\psi\rangle$ is given by

$$\psi(x) \overset{def}{=} \langle 0|\phi(x)|\psi\rangle\,,$$

where $\phi(x)$ is the scalar field, then the wave function for the state you get by applying the momentum operator to the original state is given by

$$\langle 0|\phi(x)\vec{P}|\psi\rangle = -i\vec{\nabla}\psi(x)\,.$$

(Just in case: The vacuum state $|0\rangle$ is, among other things, a state of definite momentum with momentum zero.)

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$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \partial^\mu\phi\,\partial^\nu\phi \tag{10}$$

$$P^i = \int d^3x\; T^{0i} = \int d^3x\;\partial^0\phi\;\partial^i\phi \tag{11}$$

The vacuum is an eigenstate of $\vec{P}$ with eigenvalue 0,

$$\langle 0|\phi(x)P^i|\psi\rangle = \langle 0|\phi(x)P^i - P^i\phi(x)|\psi\rangle = \langle 0|\,[\phi(x),P^i]\,|\psi\rangle \tag{12}$$

P^i is time independent, so can choose $x^0 = y^0$ in the next equation,

$$\begin{aligned}
\langle 0|\,[\phi(x),P^i]\,|\psi\rangle &= \langle 0|\int d^3y\;\left[\phi(x),\partial^0\phi(y)\;\partial^i\phi(y)\right]|\psi\rangle & (13)\\
&= \langle 0|\int d^3y\;\left(\left[\phi(x),\partial^0\phi(y)\right]\partial^i\phi(y) + \partial^0\phi(y)\left[\phi(x),\partial^i\phi(y)\right]\right)|\psi\rangle & (14)\\
&= \langle 0|\int d^3y\;\left(i\delta^3(\vec{x}-\vec{y})\partial^i\phi(y) + \text{zero}\right)|\psi\rangle & (15)\\
&= \langle 0|\,i\partial^i\phi(x)\,|\psi\rangle & (16)\\
&= -i\frac{\partial}{\partial x^i}\psi(x) & (17)
\end{aligned}$$