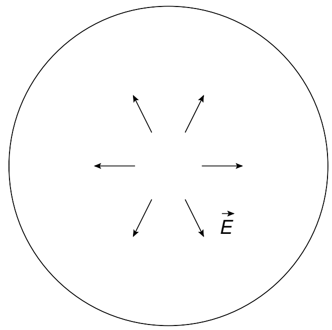
Physics 611 **Homework #4** **handed out 03 November 2016**
due 10 November 2016

1. For a cylindrical waveguide, the lowest TM mode, the TM_{01}, exhibits a purely outward (or purely inward, depending on the time) electric field in the radial direction. This is very peculiar given that there is no free charge within the waveguide.



(cross section in x–y plane)

Show for this mode, that when thinking properly in 3D, the total flux through a surface surrounding a point on the axis of the waveguide is in fact zero.

As a reminder, you should have in your notes that for the TM_{01} mode,

$$E_z = E_0\, J_0(\gamma\rho)\, e^{i(kz-\omega t)},$$
$$E_\rho = -\frac{ik}{\gamma}\, E_0\, J_1(\gamma\rho)\, e^{i(kz-\omega t)} \quad \text{(corrected)},$$
$$E_\phi = 0.$$

....

Say the point of interest on axis is between z_1 and z_2 ($z_2 > z_1$).
Integrate over a closed cylindrical surface of radius r ($r <$ waveguide radius) with endcaps at z_1 and z_2.
The total flux through the surface is,

$$\oint \vec{E}\cdot d\vec{a} = \int_{\text{curved surface}} E_\rho\, 2\pi r\, dz + \int_{(z_2)} E_z\, 2\pi\rho\, d\rho - \int_{(z_1)} E_z\, 2\pi\rho\, d\rho. \tag{1}$$

Omitting writing the time dependence,

$$\int_{\text{curved surface}} E_\rho\, 2\pi r\, dz = -\frac{2\pi i k r E_0}{\gamma} J_1(\gamma\rho) \int_{z_1}^{z_2} e^{ikz}\, dz = -\frac{2\pi r E_0}{\gamma} J_1(\gamma\rho)\left(e^{ikz_2} - e^{ikz_1}\right). \tag{2}$$

Also,

$$\int_{(z_2)} E_z\, 2\pi\rho\, d\rho = 2\pi E_0 e^{ikz_2} \int_0^r J_0(\gamma\rho)\rho\, d\rho = \frac{2\pi E_0}{\gamma} e^{ikz_2} \int_0^r d\rho\, \frac{d}{d\rho}\left(\rho J_1(\gamma\rho)\right) = \frac{2\pi r E_0}{\gamma} J_1(\gamma r) e^{ikz_2}, \tag{3}$$

using the theorem $d[xJ_1(x)]/dx = xJ_0(x)$. Similarly,

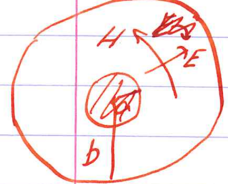
$$\int_{(z_1)} E_z\, 2\pi\rho\, d\rho = \frac{2\pi r E_0}{\gamma} J_1(\gamma r) e^{ikz_1}. \tag{4}$$

Adding up,

$$\boxed{\oint \vec{E}\cdot d\vec{a} = 0.} \tag{5}$$

2. Jackson problem 8.2.

Jackson 8.2. Coaxial waveguide.
Inside radius a, outside radius b.
metal conductivity σ, skin depth δ.
TEM mode.
"Section 8.1 applies."



a) May solve for $\vec{H} = H_0 \hat{\varphi} \frac{a}{\rho} = \vec{H}_t$

Also have $ikE_t + i\omega \hat{z} \times B_t = 0$

$$E_t = -\frac{\omega}{k} \hat{z} \times \mu H_t = \hat{\rho} \frac{\omega \mu}{k} H_0 \frac{a}{\rho}$$

$$= \hat{\rho} \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho}$$

Poynting vector $\vec{S} = \vec{E} \times \vec{H} = \hat{z} \sqrt{\frac{\mu}{\epsilon}} \frac{a^2}{\rho^2} |H_0|^2 = \frac{\text{Power}}{\text{Area}}$

Power (instantaneous) $P_{inst.} = \int \vec{S} \cdot d\vec{a} = \int_a^b 2\pi\rho\, d\rho \sqrt{\frac{\mu}{\epsilon}} \frac{a^2}{\rho^2} |H_0^2|$

$$= 2\pi a^2 \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \ln \frac{b}{a}$$

Time average power $\boxed{P = \langle P \rangle = \sqrt{\frac{\mu}{\epsilon}}\, \pi a^2 |H_0|^2 \ln \frac{b}{a}}$

b) Above is at some starting z. Now calculate power loss and power attenuation.
From the text $\frac{dP_{loss}}{da} = \frac{1}{2\sigma\delta} |H_{\parallel}|^2$ (time averaged, where ($H_{\parallel}$ is peak field at (relevant surface.)

inside: $\frac{dP_{loss}}{da} = \frac{1}{2\sigma\delta} |H_0^2|$

$da = 2\pi a\, dz$

$\therefore \frac{dP_{loss}}{dz} = \frac{\pi a}{\sigma\delta} |H_0^2|$

outside: $\frac{dP_{loss}}{dz} = \frac{\pi b}{\sigma\delta} \cdot \frac{a^2}{b^2} |H_0|^2 = \frac{\pi a^2}{\sigma\delta} \frac{|H_0^2|}{b}$

Total: $-\frac{dP_{loss}}{dz} = \frac{dP}{dz} = -\frac{\pi a^2}{\sigma\delta} |H_0^2| \left(\frac{1}{a} + \frac{1}{b}\right)$

$$\frac{dP}{dz} = -\frac{\pi a^2}{\sigma\delta}\frac{P}{\pi a^2 \ln\frac{b}{a}}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$= -\frac{1}{\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\frac{b}{a}}P \equiv -2\gamma P.$$

Hence $\boxed{P = P_0\, e^{-2\gamma z}}$ and γ is as stated in the problem.

c) $Z_0 \equiv V/I$.

(Ampère) $\bar{H} = \hat{\phi}\frac{I}{2\pi\rho}$

$$\bar{E} = -\frac{\omega\mu}{k}\hat{z}\times\vec{H} = \hat{\rho}\sqrt{\frac{\mu}{\varepsilon}}|\vec{H}| = \hat{\rho}\sqrt{\frac{\mu}{\varepsilon}}\frac{I}{2\pi\rho}$$

$$V = \int_a^b E_\rho\, d\rho = \sqrt{\frac{\mu}{\varepsilon}}\frac{I}{2\pi}\ln\frac{b}{a}$$

$$\boxed{Z_0 = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a}}$$

d) resistance per unit length, $\frac{dP}{dz} = -2\gamma P = -\langle I^2\rangle\frac{dR}{dz} = -\frac{1}{2}I^2\frac{dR}{dz}$

$\langle\ \rangle$ time average.

$$P = \sqrt{\frac{\mu}{\varepsilon}}\pi a^2 \underbrace{(H_0)^2}_{\text{field at inside surface}} \ln\frac{b}{a} = \sqrt{\frac{\mu}{\varepsilon}}\pi a^2\frac{I^2}{4\pi^2 a^2}\ln\frac{b}{a}$$

$$P = \frac{1}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}I^2\ln\frac{b}{a}$$

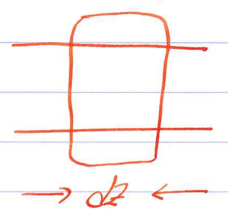
$$\therefore\quad \frac{dR}{dz} = \frac{4P}{I^2}\gamma = \frac{1}{\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a}\cdot\frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\frac{b}{a}}$$

$$\boxed{\frac{dR}{dz} = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)}$$ (Jackson calls this "R".)

d, part 2) inductance per unit length.

Use $\frac{dV}{dz} = -\frac{dL}{dz}\frac{dI}{dt}$ and Faraday's law.

Side view, with loop for flux integral:



The loop extends far enough to catch all the ~~flux~~ magnetic field that penetrates the conductors.

$$\mathcal{E} = \frac{dV}{dz} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \qquad |d\vec{a}| = d\rho\, dz$$

Here, $$\frac{dV}{dz} = -\frac{d}{dt}\int B_\phi\, d\rho$$

~~Between~~ $$= -\frac{d}{dt}\left\{ \int_a^b B_\phi\, d\rho + \underset{\substack{(\text{inside})\\(\rho<a)}}{\int_0^\infty B_\phi\, d\xi} + \underset{\substack{\text{outside,}\\ \rho>b.}}{\int_0^\infty B_\phi\, d\xi} \right\}$$

$$\int_a^b B_\phi\, d\rho = \frac{\mu I}{2\pi}\int_a^b \frac{d\rho}{\rho} = \frac{\mu}{2\pi}\ln\frac{b}{a}\cdot I$$

$$\underset{(\rho<a)}{\int_0^\infty} B_\phi\, d\xi = \int_0^\infty \frac{\mu_c I}{2\pi a} e^{-(1-i)\xi/\delta}\, d\xi = \frac{\mu_c I}{2\pi a}\frac{\delta}{1-i}$$

$$= \frac{\mu_c \delta}{4\pi a}(1+i)\, I$$

Only keep real part : $\text{Re} \underset{(\rho<a)}{\int_0^\infty} B_\phi\, d\xi = \frac{\mu_c\delta}{4\pi a} I$

Same for outside conductor, with $a \to b$.

$$\therefore \quad \boxed{\frac{dL}{dz} = \frac{\mu}{2\pi}\ln\frac{b}{a} + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)}$$ (Jackson calls this "L.")

3. (a) Find for electric dipole radiation in the far zone, also known as the radiation zone and characterized by $r/\lambda \gg 1$, the ratio

$$\frac{|c\vec{B}|}{|\vec{E}|}.$$

(b) Find the same ratio for the near zone, also known as the static zone and characterized by $r/\lambda \ll 1$.

(c & d) Find the same ratio for magnetic dipole radiation in the far and near zones.

. . . .

For electric radiation,

$$\begin{aligned} \vec{H} &= \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\left(\hat{n}\times\vec{p}\right)\left(1-\frac{1}{ikr}\right), \\ \vec{E} &= \frac{1}{4\pi\epsilon_0}\left\{k^2\left(\hat{n}\times\vec{p}\right)\times\hat{n}\frac{e^{ikr}}{r} + e^{ikr}\left(3\hat{n}\left(\hat{n}\cdot\vec{p}\right)-\vec{p}\right)\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\right\}. \end{aligned} \tag{6}$$

(a) In the radiation zone ($r\to\infty$),

$$\begin{aligned} \vec{B} &= \mu_0\vec{H} = \frac{\mu_0 ck^2}{4\pi}\frac{e^{ikr}}{r}\left(\hat{n}\times\vec{p}\right), \\ \vec{E} &= \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left(\hat{n}\times\vec{p}\right)\times\hat{n}. \end{aligned} \tag{7}$$

Hence

$$\frac{|c\vec{B}|}{|\vec{E}|} = \epsilon_0\mu_0 c^2 = 1. \tag{8}$$

(b) In the near zone ($r\to 0$),

$$\begin{aligned} \vec{B} &= \frac{i\mu_0 ck}{4\pi}\frac{e^{ikr}}{r^2}\left(\hat{n}\times\vec{p}\right), \\ \vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^3}\left(3\hat{n}\left(\hat{n}\cdot\vec{p}\right)-\vec{p}\right), \end{aligned} \tag{9}$$

so that,

$$\frac{|c\vec{B}|}{|\vec{E}|} = \epsilon_0\mu_0 c^2 \times \mathcal{O}(kr) \to 0. \tag{10}$$

(c) and (d), the magnetic cases, are similar, with $\vec{E}\leftrightarrow \pm c\vec{B}$ and $\vec{p}\to\vec{m}/c$.
(c) is as above, and
(d)

$$\frac{|c\vec{B}|}{|\vec{E}|} = \epsilon_0\mu_0 c^2 \times \mathcal{O}\left(\frac{1}{kr}\right) \to \infty. \tag{11}$$