Physics 722 Quantum Field Theory

homework 3 due 23 March 2017

1. Bjorken and Drell, Ch. 19, problem 1, which is:

Prove the generalized Ward identity [the Ward-Takahashi identity] by forming the three-fold vacuum expectation value

$$\langle 0| T\left(\psi(x)\bar{\psi}(y) j_\mu(z)\right)|0\rangle$$

and using current conservation and the field equations.

(The current $j_\mu(z)$ is as usual $\bar{\psi}(z)\gamma_\mu\psi(z)$.)

. . . .

The basic object is

$$\langle 0|T\psi(x)\bar{\psi}(y) j_\mu(z)|0\rangle = \langle 0|T\psi(x)\bar{\psi}(y)\bar{\psi}(z)\gamma_\mu\psi(z)|0\rangle\,. \tag{1}$$

Be aware that the fields are fields with all interactions included. Also be aware that the equal time canonical commutation relations of the fields only depend on the rules of quantum mechanics, and so work for the interacting fields. There is some choice regarding what steps to take first.

• I will start with using current conservation.

$$\begin{aligned}\frac{\partial}{\partial z_\mu}\langle 0|T\psi(x)\bar{\psi}(y) j_\mu(z)|0\rangle = \frac{\partial}{\partial z_\mu}\langle 0|\ &\theta(x_0-y_0)\theta(y_0-z_0)\ \psi(x)\bar{\psi}(y) j_\mu(z)\\ &+\theta(x_0-z_0)\theta(z_0-z_0)\ \psi(x) j_\mu(z)\bar{\psi}(y)\\ &+\ 4 \text{ more terms} \qquad |0\rangle.\end{aligned} \tag{2}$$

The derivatives will act on the theta-functions, as well as on the current. On the current it will give $\partial_z^\mu j_\mu = 0$, and on the theta-functions it will give delta-functions. One will find,

$$\begin{aligned}\partial_z^\mu \langle 0|T\psi(x)\bar{\psi}(y) j_\mu(z)|0\rangle &= \delta(y_0-z_0)\,\langle 0|\,T\psi(x)\left[j_0(z),\bar{\psi}(y)\right]\,|0\rangle\\ &\quad -\delta(x_0-z_0)\,\langle 0|\,T\left[\psi(x), j_0(z)\right]\bar{\psi}(y)\,|0\rangle\,.\end{aligned} \tag{3}$$

Then, with "ET" standing for "equal time,"

$$\begin{aligned}\left[j_0(z),\psi(y)^\dagger\right]_{\rm ET} &= \left[\psi^\dagger(z)\psi(z),\psi(y)^\dagger\right]_{\rm ET} = \psi^\dagger(z)\left\{\psi(z),\psi(y)^\dagger\right\}_{\rm ET} = \psi^\dagger(z)\delta^3(z-y),\\ \left[\psi(x), j_0(z)\right]_{\rm ET} &= \psi(z)\delta^3(x-z).\end{aligned} \tag{4}$$

Hence,

$$\partial_z^\mu \langle 0|T\psi(x)\bar{\psi}(y)j_\mu(z)|0\rangle = \delta^4(y-z)\langle 0|\,T\psi(x)\bar{\psi}(y)\,|0\rangle - \delta^4(x-z)\langle 0|\,T\psi(x)\bar{\psi}(y)\,|0\rangle\,, \quad (5)$$

or

$$\boxed{\partial_z^\mu \langle 0|T\psi(x)\bar{\psi}(y)j_\mu(z)|0\rangle = iS_F'(x-z)\,\delta^4(y-z) - iS_F'(z-y)\,\delta^4(x-z).} \quad (6)$$

- To get factors of "i", *etc.*, work out 3-point function in LO.

$$\begin{aligned}\langle 0|T\psi(x)\bar{\psi}(y)\,\bar{\psi}(z)\gamma_\mu\psi(z)|0\rangle &= \langle 0|T\psi(x)\,\bar{\psi}(z)\gamma_\mu\psi(z)\,\bar{\psi}(y)|0\rangle \\ = \langle 0|T\psi(x)\,\bar{\psi}(z)|0\rangle\,\gamma_\mu\,\langle 0|\psi(z)\,\bar{\psi}(y)|0\rangle &= iS_F(x-z)\,\gamma_\mu\,iS_F(z-y)\end{aligned} \quad (7)$$

Using momentum space,

$$\begin{aligned}S_F(x-z) &= \int (d^4k)e^{-ik(x-z)}S_F(k), \\ S_F(p') &= \int (d^4x)e^{ip'(x-z)}S_F(x-z), \\ S_F(p) &= \int (d^4x)e^{ip(z-y)}S_F(z-y).\end{aligned} \quad (8)$$

Hence,

$$\int (d^4x)(d^4y)e^{ip'(x-z)}e^{ip(z-y)}\,\langle 0|T\psi(x)\bar{\psi}(y)j_\mu(z)|0\rangle = iS_F(p')\,\gamma_\mu\,iS_F(p). \quad (9)$$

- General case definition,

$$\boxed{\int (d^4x)(d^4y)e^{ip'(x-z)}e^{ip(z-y)}\,\langle 0|T\psi(x)\bar{\psi}(y)j_\mu(z)|0\rangle = iS_F'(p')\,\Gamma_\mu\,iS_F'(p).} \quad (10)$$

- Finally,

$$\begin{aligned}&\int (d^4x)(d^4y)e^{ip'(x-z)}e^{ip(z-y)}\;\partial_z^\mu\,\langle 0|T\psi(x)\bar{\psi}(y)j_\mu(z)|0\rangle = i(p'-p)^\mu \int \ldots \\ &= i^3 S_F'(p')\,(p'-p)^\mu\Gamma_\mu\,S_F'(p) \\ &= \int (d^4x)(d^4y)e^{ip'(x-z)}e^{ip(z-y)}\,\left[iS_F'(x-z)\,\delta^4(y-z) - iS_F'(z-y)\,\delta^4(x-z)\right] \\ &= iS_F'(p') - iS_F'(p).\end{aligned} \quad (11)$$

Thus,

$$S_F'(p')\,(p'-p)^\mu\Gamma_\mu\,S_F'(p) = S_F'(p) - S_F'(p'). \quad (12)$$

Thus,

$$\boxed{(p'-p)^\mu\Gamma_\mu = {S_F'}^{-1}(p') - {S_F'}^{-1}(p).} \quad (13)$$

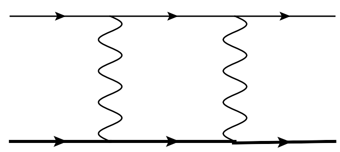
2. Suppose the QED Lagrangian contained an actual magnetic moment interaction, as

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi - e_0\bar{\psi}\gamma_\mu\psi\, A^\mu - \frac{\kappa_0}{2m_0}\bar{\psi}\sigma_{\mu\nu}\psi\, F^{\mu\nu}.$$

Give the superficial degree of divergence for Feynman diagrams in this theory in 4D in terms of the numbers of external legs and, if necessary, the number of vertices in the diagram. How many primitively divergent diagrams are there? For the sake of simplicity, you may let $e_0 = 0$.

Look up again, or recall, the meaning of renormalizable, superrenormalizable, and non-renormalizable, and state which of these QED with an intrinsic magnetic moment term is.

What is the degree of divergence of the box diagram,



where each vertex is the κ_0 vertex above?

. . . .

The κ_0 vertex has a derivative (within $F^{\mu\nu}$), and hence a power of momentum in momentum space. Say all the vertices are κ_0 vertices.

$$D = 4L - 2P_\gamma - P_e + p(V - N_\gamma) \tag{14}$$

where $p = 1$ for the κ_0 vertex, and the standard result can be checked using $p = 0$. P_γ and P_e are the number of internal photon and electron propagators, and N_γ and N_e are the number of external photon and electron lines.

$$L = P_\gamma + P_e - V + 1 \tag{15}$$

$$V = 2P_\gamma + N_\gamma \tag{16}$$

$$2V = 2P_e + N_e \tag{17}$$

Substituting,

$$\boxed{D = 4 - N_\gamma - \frac{3}{2}N_e + p(V - N_\gamma)} \tag{18}$$

The are an infinite number of primitive divergences.

The theory is non-renormalizable.

The box diagram has

$$D = 4 - 0 - 6 + (4 - 0) = 2. \tag{19}$$

(For ordinary QED it would have $D = -2$.)