Physics 722 — Quantum Field Theory

homework 2 — due 23 February 2017

1. For Grassmann variables ξ and θ, show that one has a Fourier transform pair

$$g(\xi) \;\; = \int d\theta\, e^{i\xi\theta}\, f(\theta)\,, \tag{1}$$

$$f(\theta) \; = i \int d\xi\, e^{-i\xi\theta}\, g(\xi)\,. \tag{2}$$

I.e., using the first equation to define $g(\xi)$, show that the second equation is true.

. . . .

Expand both $f(\theta)$ and $e^{i\xi\theta}$,

$$\begin{aligned} f(\theta) &= a + b\theta, \\ e^{i\xi\theta} &= 1 + i\xi\theta, \end{aligned} \tag{3}$$

and do the first integral, remembering $\int d\theta\, \theta = 1$,

$$g(\xi) = \int d\theta\, (1 - i\theta\xi)(a + b\theta) = \int d\theta\, \left(a + b\theta - ia\theta\xi\right) = b - ia\xi\,. \tag{4}$$

Then

$$f(\theta) \stackrel{?}{=} i \int d\xi\, (1 - i\xi\theta)(b - ia\xi) = i \int d\xi\, \left(b - ia\xi - ib\xi\theta\right) = a + b\theta\,. \tag{5}$$

Works!

2. Calculate the superficial degree of divergence in 4D for Yukawa theory, which is a theory of interacting fermions and scalars,

$$\mathcal{L} = \mathcal{L}_{0\psi} + \mathcal{L}_{0\phi} - g\bar{\psi}\psi\phi\,.$$

Look up the meaning of renormalizable, superrenormalizable, and non-renormalizable, and state which of these Yukawa theory is in 4D.

. . . .

Let P_f = number of internal fermion lines (fermion propagators) and P_s = number of internal scalar lines. Then if L is the number of independent loops, the superficial degree of divergence is

$$D = 4L - P_f - 2P_s\,. \tag{6}$$

If V is the number of vertices, the number in independent loops is the number of independent internal momenta, which is

$$L = P_f + P_s - (V - 1)\,. \tag{7}$$

Hence,

$$D = 4 - 4V + 3P_f + 2P_s\,. \tag{8}$$

Noticing that each vertex has the ends of two fermion lines and the end of one scalar line,

$$\begin{aligned} 2V &= 2P_f + N_f\,, \\ V &= 2P_s + N_s\,, \end{aligned} \tag{9}$$

where N_f is the number of external fermion lines and N_s is the number of external scalar lines. Using this to substitute for the number of propagators,

$$D = 4 - 4V + 3V - \frac{3}{2}N_f + V - N_s, \tag{10}$$

or

$$\boxed{D = 4 - \frac{3}{2}N_f - N_s\,.} \tag{11}$$

Renormalizable.

Same counting as QED.

3. Calculate the superficial degree of divergence for scalar electrodynamics (charged pions and photons) in d dimensions, *i.e.,* where the integrations are $d^d k$ and the propagators have the same powers of momentum as in 4D. Show that in 4D that the superficial degree of divergence of a Feynman diagram depends only on the numbers of external legs.

(The Lagrangian is

$$\begin{aligned} \mathcal{L} &= [(\partial - ieA)_\mu \phi]^* (\partial - ieA)^\mu \phi - m^2 \phi^* \phi \\ &= \mathcal{L}_0 - ie\phi^* \left(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu \right) \phi A^\mu + e^2 \phi^* \phi A_\mu A^\mu \ , \end{aligned}$$

where the interaction terms give Feynman rules $-ie(p+p')_\mu$ and $2ie^2 g_{\mu\nu}$, respectively.)

. . . .

Noting a power of momentum at each vertex that has three lines entering,

$$D = dL + V_3 - 2P_\pi - 2P_\gamma \, . \tag{12}$$

There are two types of vertices, with 3 lines entering or 4 lines entering, with numbers V_3 and V_4, and

$$L = P_\pi + P_\gamma - (V_3 + V_4 - 1) \, , \tag{13}$$

and

$$D = d - (d-1)V_3 - dV_4 + 2P_\pi + 2P_\gamma \, . \tag{14}$$

Further,

$$\begin{aligned} 2V_3 + 2V_4 &= 2P_\pi + N_\pi \, , \\ V_3 + 2V_4 &= 2P_\gamma + N_\gamma \, . \end{aligned} \tag{15}$$

Hence,

$$D = d - (d-1)V_3 - dV_4 + 2V_3 + 2V_4 - N_\pi + V_3 + 2V_4 - N_\gamma \, , \tag{16}$$

or

$$\boxed{D = d - (d-4)V_3 - (d-4)V_4 - N_\pi - N_\gamma \, .} \tag{17}$$

In four dimensions this is $D = 4 - N_\pi - N_\gamma$.