Physics 611 **Homework #2** **handed out 15 September 2016**
due 22 September 2016

1. For an electric monopole (point electric charge) with charge e and a magnetic monopole of charge g separated from it by distance R, find the angular momentum stored in the fields. Requirement: use a method noticeably different from the one shown in Jackson.

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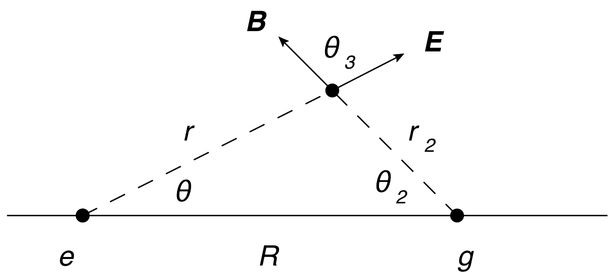
Diagram:



Let $\vec{g}$ = momentum density in fields; out of page at point where $\vec{E}$ and $\vec{B}$ fields are indicated in diagram.

$$|\vec{g}| = \varepsilon_0 |\vec{E} \times \vec{B}| = \varepsilon_0 \, \frac{1}{4\pi\varepsilon_0} \frac{e}{r^2} \, \frac{1}{4\pi} \frac{g}{r_2^2} \, \sin\theta_3 \,. \tag{1}$$

Let $\vec{\ell}$ be the angular momentum density.

$$\ell_z = (r \sin\theta)\, |\vec{g}| = \frac{eg}{16\pi^2} \frac{1}{r r_2^2} \sin\theta \sin\theta_3 \,. \tag{2}$$

By law of sines,

$$\frac{\sin\theta_3}{R} = \frac{\sin\theta}{r_2}\,, \qquad \text{so that} \qquad \ell_z = \frac{eg}{16\pi^2} \frac{R \sin^2\theta}{r r_2^3}, \tag{3}$$

and

$$L_z = \int d^3x \; \ell_z = \frac{eg}{8\pi} \int dr \int d(\cos\theta) \frac{r R \sin^2\theta}{r_2^3}. \tag{4}$$

Use $r_2^2 = R^2 + r^2 - 2rR\cos\theta = R^2(1 + s^2 - 2s\cos\theta)$ for $s \equiv r/R$. Then

$$L_z = \frac{eg}{8\pi} \int_0^\infty ds \int_{-1}^1 d(\cos\theta) \frac{s \sin^2\theta}{(1 + s^2 - 2s\cos\theta)^{3/2}} = \frac{eg}{8\pi} \int_0^\infty ds \int_{-1}^1 d(\cos\theta) \frac{2\cos\theta}{(1 + s^2 - 2s\cos\theta)^{1/2}}\,, \tag{5}$$

where the second step involves an integration by parts. Then with $\cos\theta = P_1(\cos\theta)$ and recognizing $(1 + s^2 - 2s\cos\theta)^{-1/2}$ as the generator of Legendre polynomials,

$$\begin{aligned} L_z &= \frac{eg}{4\pi} \bigg\{ \int_0^1 ds \int_{-1}^1 d(\cos\theta)\, P_1(\cos\theta) \sum s^n P_n(\cos\theta) \\ &\qquad + \int_1^\infty ds \int_{-1}^1 d(\cos\theta)\, P_1(\cos\theta) \sum s^{-(n+1)} P_n(\cos\theta) \bigg\} \\ &= \frac{eg}{4\pi} \left\{ \int_0^1 ds \cdot \frac{2}{3} s + \int_1^\infty ds \cdot \frac{2}{3} \frac{1}{s^2} \right\} = \frac{eg}{4\pi} \left\{ \frac{1}{3} + \frac{2}{3} \right\} \end{aligned} \tag{6}$$

Thus

$$\boxed{L_z = \frac{eg}{4\pi}} \tag{7}$$

2. For a plane wave moving in the $\hat{z}$ direction, the electric field may be written as

$$\vec{E}(\vec{x}, t) = \text{Re}\left[(E_1\hat{x} + E_2\hat{y})\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right] .$$

a) For $z = 0$, find the x and y components of the electric field for the case E_1 and E_2 are 90° out of phase and E_2 has one third the magnitude of E_1. Sketch the evolution of $\vec{E}$ with time over one cycle.

b) Again for $z = 0$, find the x and y components of the electric field for the case E_1 and E_2 are 45° out of phase and E_2 has the same magnitude as E_1. Again, sketch the evolution of $\vec{E}$ with time over one cycle.

(You may make the sketches by hand, or using Mathematica or equivalent.)

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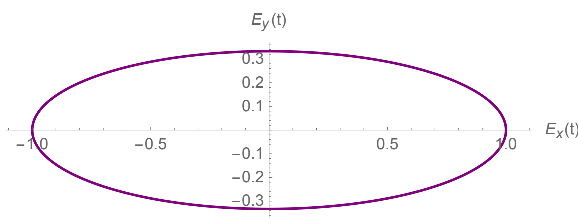
Let $E_1 = a_1$ be real for ease of presentation, and let generally $E_2 = a_2 e^{i\phi}$ and let $z = 0$.

$$\begin{aligned}\vec{E} &= \text{Re}\left[a_1\hat{x}\, e^{-i\omega t} + a_2\,\hat{y}\, e^{-i(\omega t-\phi)}\right]\\ E_x &= a_1\cos(\omega t)\\ E_y &= a_2\cos(\omega t - \phi)\end{aligned} \tag{8}$$

(a)

$$\begin{aligned}E_x &= a_1\cos(\omega t)\\ E_y &= 0.333\, a_1\cos(\omega t - 90^\circ) = 0.333\, a_1\sin(\omega t)\end{aligned} \tag{9}$$

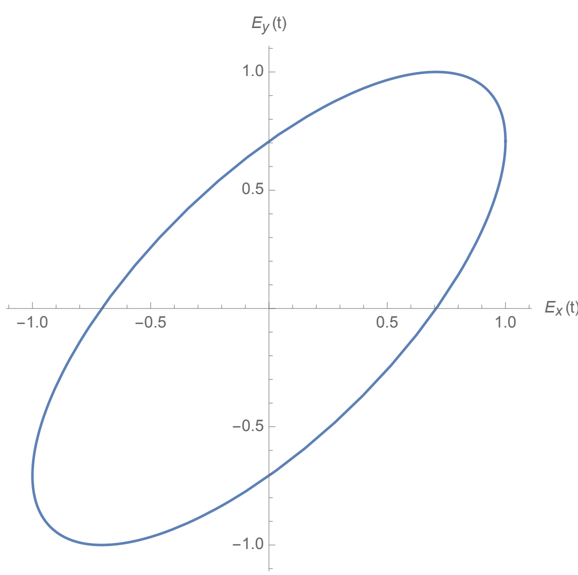
Diagram for $a_1 = 1$,



(b)

$$\begin{aligned}E_x &= a_1\cos(\omega t)\\ E_y &= a_1\cos(\omega t - 45^\circ)\end{aligned} \tag{10}$$

Diagram again for $a_1 = 1$,



3. Jackson problem 7.27.
a) From

$$\vec{L} = \frac{1}{\mu_0 c^2}\int d^3x\ \vec{x}\times(\vec{E}\times\vec{B}), \tag{11}$$

show

$$\vec{L} = \frac{1}{\mu_0 c^2}\int d^3x \left[\vec{E}\times\vec{A} + \vec{E}\cdot(\vec{x}\times\vec{\nabla})\vec{A}\right]. \tag{12}$$

b)For

$$\vec{A}(\vec{x},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[\varepsilon_\lambda(\vec{k})a_\lambda(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.}\right], \tag{13}$$

show

$$\langle\vec{L}_{\text{spin}}\rangle = \frac{2}{\mu_0 c}\int \frac{d^3k}{(2\pi)^3}\vec{k}\left[\left|a_+(\vec{k})\right|^2 - \left|a_-(\vec{k})\right|^2\right]. \tag{14}$$

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a)

$$\begin{aligned}\vec{L} &= \frac{1}{\mu_0 c^2}\int d^3x\ \vec{x}\times\left(\vec{E}\times(\vec{\nabla}\times\vec{A})\right)\\ &= \frac{1}{\mu_0 c^2}\int d^3x\ \vec{x}\times\left(\vec{E}\cdot(\vec{\nabla})\vec{A} - (\vec{E}\cdot\vec{\nabla})\vec{A}\right)\\ &= \frac{1}{\mu_0 c^2}\int d^3x\left(\vec{E}\cdot(\vec{x}\times\vec{\nabla})\vec{A} - \vec{x}\times(\vec{E}\cdot\vec{\nabla})\vec{A}\right)\end{aligned} \tag{15}$$

The first term is directly one of the terms we want. Integrate by parts on the other term. Use $\vec{\nabla}\cdot\vec{E} = 0$, since we are in free space. For the other contribution, use $\nabla_i x_j = \delta_{ij}$ to effectively substitute $\vec{E}$ for $\vec{x}$ (or write the term out in components). Result:

$$\vec{L} = \frac{1}{\mu_0 c^2}\int d^3x\left[\vec{E}\cdot(\vec{x}\times\vec{\nabla})\vec{A} + \vec{E}\times\vec{A}\right]. \tag{16}$$

b)

$$\begin{aligned}\vec{A}(\vec{x},t) &= \sum_\lambda\int\frac{d^3k}{(2\pi)^3}\left[\hat{\varepsilon}_\lambda(\vec{k})a_\lambda(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.}\right] = \text{Re}\left[\vec{A}_0(\vec{x})e^{-i\omega t}\right],\\ \vec{E}(\vec{x},t) &= -\frac{\partial\vec{A}(\vec{x},t)}{\partial t} = \text{Re}\left[i\omega\vec{A}_0(\vec{x})e^{-i\omega t}\right],\end{aligned} \tag{17}$$

for

$$\vec{A}_0 = 2\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\hat{\varepsilon}_\lambda(\vec{k})a_\lambda(\vec{k})e^{i\vec{k}\cdot\vec{x}}. \tag{18}$$

The problem asks for the time average of the $\vec{E}\times\vec{A}$ term.

$$\langle\vec{L}_{\text{spin}}\rangle = \frac{1}{\mu_0 c^2}\int d^3x\ \frac{1}{2}\text{Re}\left(\vec{E}_0\times\vec{A}_0^*\right). \tag{19}$$

Do the d^3x integral to obtain a momentum δ-function, leaving only one set of momentum integrals.

$$\langle \vec{L}_{\rm spin} \rangle = \frac{2}{\mu_0 c^2} \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \text{Re}\left[i\omega\, a_\lambda(\vec{k}) a^*_{\lambda'}(\vec{k})\ \hat{\varepsilon}_\lambda(\vec{k}) \times \hat{\varepsilon}^*_{\lambda'}(\vec{k}) \right]. \tag{20}$$

Show that

$$\hat{\varepsilon}_\lambda(\vec{k}) \times \hat{\varepsilon}^*_{\lambda'}(\vec{k}) = -i\lambda\hat{k}\delta_{\lambda,\lambda'}. \tag{21}$$

Then

$$\langle \vec{L}_{\rm spin} \rangle = \frac{2}{\mu_0 c^2} \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3}\ \omega\lambda\hat{k} \left| a_\lambda(\vec{k}) \right|^2, \tag{22}$$

or

$$\boxed{\langle \vec{L}_{\rm spin} \rangle = \frac{2}{\mu_0 c} \int \frac{d^3k}{(2\pi)^3}\, \vec{k} \left(\left| a_+(\vec{k}) \right|^2 - \left| a_+(\vec{k}) \right|^2 \right).} \tag{23}$$