Physics 611 **Homework #1 with answers** **handed out 01 September 2016**

1. In Heaviside-Lorentz units the force between two point charges can be written as (in magnitude)

$$F = \frac{1}{4\pi}\frac{|q_1 q_2|}{r^2},$$

and the Lorentz force is

$$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c} \times \vec{B}.$$

a) Write Maxwell's equations in Heaviside-Lorentz units.
b) The energy density in the electric and magnetic fields in SI units is

$$u = \frac{\epsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2.$$

Rewrite the energy density formula in gaussian and in Heaviside-Lorentz units.
c) Work out what magnetic field in gaussian units corresponds to a magnetic field of 1 Tesla in SI units.

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a) Looking at the above Coulomb and Lorentz force laws, to get from SI to HL units,

$$\frac{1}{\epsilon_0} \to 1, \quad \mu_0 = \frac{1}{\epsilon_0 c^2} \to \frac{1}{c^2}, \quad B \to \frac{1}{c}B\,. \tag{1}$$

The SI Maxwell equations,

$$\begin{aligned} \nabla\cdot E &= \frac{\rho}{\epsilon_0} & \qquad \nabla\times E &= -\frac{\partial B}{\partial t} \\ \nabla\cdot B &= 0 & \qquad \nabla\times B &= \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \end{aligned} \tag{2}$$

become the HL Maxwell equations,

$$\begin{aligned} \nabla\cdot E &= \rho & \qquad \nabla\times E &= -\frac{1}{c}\frac{\partial B}{\partial t} \\ \nabla\cdot B &= 0 & \qquad \nabla\times B &= \frac{1}{c}J + \frac{1}{c}\frac{\partial E}{\partial t} \end{aligned} \tag{3}$$

b) Using the same substitutions, the energy density becomes in HL units

$$u = \frac{1}{2}\left(E^2 + B^2\right). \tag{4}$$

For the gaussian case, the substitutions that convert from SI to gaussian units were given in class and were

$$\frac{1}{\epsilon_0} \to 4\pi, \quad \mu_0 = \frac{1}{\epsilon_0 c^2} \to \frac{4\pi}{c^2}, \quad B \to \frac{1}{c}B\,. \tag{5}$$

Tis leads to

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right). \tag{6}$$

c) An alternative to calling the gaussian unit of charge a statcoulomb is to call it an "electrostatic unit" or esu. In class we worked out that $1\ \mathrm{C} = 3 \times 10^9$ esu, and if we go back one step before the end, we could have left this as

$$1\ \mathrm{C} = 10^{-1} \frac{c}{\mathrm{cm/sec}}\ \mathrm{esu}\,, \tag{7}$$

where c is the speed of light.
For the magnetic field, consider that a 1 Tesla field upon a 1 C change moving at 1 m/s (in an appropriate direction) gives a 1 N force. From $\vec{F} = q(\vec{v}/c) \times \vec{B}$, this is also

$$B = \frac{F}{v}\frac{c}{q} = \frac{10^5\ \mathrm{dyne}}{10^2\ \mathrm{cm/sec}} \cdot \frac{c}{10^{-1}\frac{c}{\mathrm{cm/sec}}\mathrm{esu}} = 10^4 \frac{\mathrm{dyne}}{\mathrm{esu}} \tag{8}$$

In gaussian units, when we are measuring a B-field we choose to let "dyne/esu" be called "gauss," and so

$$1\ \mathrm{Tesla} = 10^4\ \mathrm{gauss}\,. \tag{9}$$

2. Show that for a general but suitably differentiable function $z = z(x, y)$ that

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1.$$

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Given $z = z(x, y)$, one has

$$dz = \left.\frac{\partial z}{\partial x}\right|_y dx + \left.\frac{\partial z}{\partial y}\right|_x dy\,. \tag{10}$$

Setting $dx \to 0$, one can rearrange this to

$$\left.\frac{\partial y}{\partial z}\right|_x = \frac{1}{\left.\frac{\partial z}{\partial y}\right|_x}\,, \tag{11}$$

and setting $dz = 0$ leads to

$$\left.\frac{\partial x}{\partial y}\right|_z = -\frac{\left.\frac{\partial z}{\partial y}\right|_x}{\left.\frac{\partial z}{\partial x}\right|_y}\,. \tag{12}$$

Now elementary multiplication gives the desired result.

3. ϵ exercises. Tensor ϵ_{ijk} (where each of i, j, k has allowed values $1, 2, 3$) is a totally antisymmetric tensor normalized by $\epsilon_{123} = 1$.
Notice that the ϵ-symbol is useful for writing cross products: $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k \equiv \sum_{jk} \epsilon_{ijk} A_j B_k$.
a) Convince yourself that

$$\sum_i \epsilon_{ijk} \epsilon_{ilm} \equiv \epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \,.$$

b) Show that

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) = -\nabla^2 \vec{B} + \vec{\nabla} \left(\vec{\nabla} \cdot \vec{B} \right) \,.$$

c) Show that

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \,.$$

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a) Be thou convinced.
b)

$$\begin{aligned} \left[\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) \right]_i &= \epsilon_{ijk} \nabla_j \left(\vec{\nabla} \times \vec{B} \right)_k = \epsilon_{ijk} \epsilon_{klm} \nabla_j \nabla_l B_m \\ &= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \nabla_j \nabla_l B_m = \nabla_i \left(\vec{\nabla} \cdot \vec{B} \right) - \nabla^2 B_i \\ &= \left[-\nabla^2 \vec{B} + \vec{\nabla} \left(\vec{\nabla} \cdot \vec{B} \right) \right]_i \,. \end{aligned} \tag{13}$$

c)

$$\begin{aligned} (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) &= (\vec{A} \times \vec{B})_i (\vec{C} \times \vec{D})_i = \epsilon_{ijk} \epsilon_{ilm} A_j B_k C_l D_m \\ &= \left(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \right) A_j B_k C_l D_m \\ &= A_j C_j B_k D_k - A_j D_j B_k C_k = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \,. \end{aligned} \tag{14}$$

4. Jackson problem 6.1.
a) Find solution to

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\vec{x},t) = -4\pi f(\vec{x},t)$$

for $f(\vec{x}',t') = \delta(x')\delta(y')\delta(t')$.
b) Same for $f(\vec{x}',t') = \delta(x')\delta(t')$.

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a) Use

$$\Psi(\vec{x},t) = \int d^3x' \, \frac{[f(\vec{x}',t')]_{\text{ret}}}{|\vec{x}-\vec{x}'|}.$$

The line of charge is the line $x' = y' = 0$. Can choose to calculate for $z = 0$ and with $x^2 + y^2 = \rho^2$.

$$|\vec{x}-\vec{x}'| \to \sqrt{\rho^2 + (z')^2},$$

and

$$\Psi(\vec{x},t) = \int_{-\infty}^{\infty} dz' \, \frac{\delta(t - \sqrt{\rho^2+(z')^2}/c)}{\sqrt{\rho^2+(z')^2}}.$$

The δ-function is non-zero for $\sqrt{\rho^2+(z')^2} = ct$, or $z' = \pm\sqrt{c^2t^2-\rho^2}$, and we note that ct has to be at least as large as ρ for this to occur. A factor $\theta(ct-\rho)$ will be tacit until the last step.

$$\Psi(\vec{x},t) = 2\int_0^{\infty} dz' \, \frac{\delta(t - \sqrt{\rho^2+(z')^2}/c}{\sqrt{\rho^2+(z')^2}} = \frac{2}{ct}\left(\frac{1}{c}\left|\frac{\partial}{\partial z'}\sqrt{\rho^2+(z')^2}\right|_{z'=\sqrt{c^2t^2-\rho^2}}\right)^{-1}$$
$$= \frac{2}{ct}\left(\frac{1}{c}\frac{z'}{\sqrt{c^2t^2-\rho^2}}\right)^{-1}_{z'=\sqrt{\rho^2+(z')^2}} = \frac{2}{ct}\frac{c\cdot ct}{\sqrt{c^2t^2-\rho^2}}$$

or

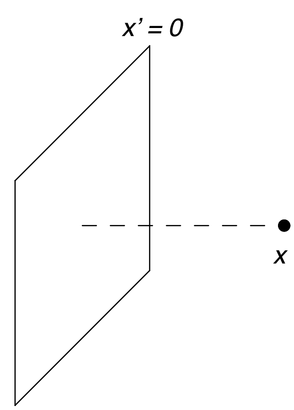
$$\boxed{\Psi(\vec{x},t) = \frac{2c\,\theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}}$$

b) Same in 1D, *i.e.,* plane of charge. The plane of charge is the $x' = 0$ plane. Find Ψ at point $y = z = 0$ and $x > 0$. Let $s^2 = (y')^2 + (z')^2$, so that

$$|\vec{x}-\vec{x}'| \to \sqrt{x^2+s^2},$$

and

$$\Psi(\vec{x},t) = \int_{-\infty}^{\infty} dy'dz' \, \frac{\delta(t-\sqrt{x^2+s^2}/c)}{\sqrt{x^2+s^2}} = \pi\int_0^{\infty} ds^2 \, \frac{\delta(t-\sqrt{x^2+s^2}/c)}{\sqrt{x^2+s^2}}$$
$$= \frac{\pi}{t}\theta(ct-x)\left(\left|\frac{\partial}{\partial s^2}\sqrt{x^2+s^2}\right|_{\sqrt{x^2+s^2}=ct}\right)^{-1}.$$



Thus

$$\boxed{\Psi(\vec{x},t) = 2\pi c\,\theta(ct-x).}$$

5. Jackson problem 6.11. (If you are uncertain about the "solar wind,", you may omit the response to the question in the last sentence of the problem.)
Plane wave on flat totally absorbing screen.
a) Show pressure on screen equals energy/volume of incoming wave.
b) For $C = 1.4\mathrm{kW/m^2}$ on sail of 1 $\mathrm{gm/cm^2}$, find maximum acceleration.

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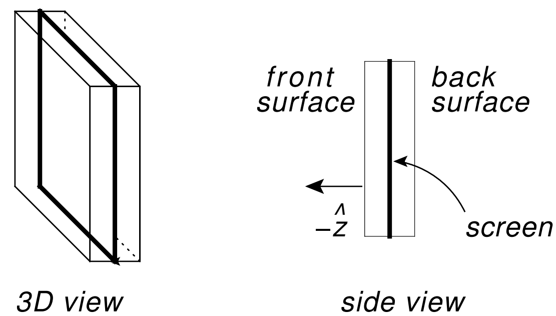
a) The force given in terms of a surface integral involving the stress tensor was derived from momentum conservation,

$$\langle \vec{F} \rangle = \oint_S da\ \hat{n} \cdot \langle \overleftrightarrow{T} \rangle\ . \tag{15}$$

For the totally absorbing screen perpendicular to the propagation direction of a wavefront, surround it by a closed surface with thin rectangular sides.



Because the screen is totally absorbing, there is no field immediately behind it, but the fields just before it are what they would have been in the absence of the screen. In addition, four sides of the surface are very thin, so the surface integral for a wave traveling in the $+z$ direction is just,

$$\langle \vec{F} \rangle = -\int_{\text{front surface}} da\ \hat{z} \cdot \langle \overleftrightarrow{T} \rangle\ , \tag{16}$$

or

$$P = -\langle T_{zz} \rangle \tag{17}$$

for the present case. With

$$T_{ij} = \epsilon_0 \left[E_i E_j - \frac{1}{2}\delta_{ij}\vec{E}^2 \right] + \frac{1}{\mu_0}\left[B_i B_j - \frac{1}{2}\delta_{ij}\vec{B}^2 \right] \tag{18}$$

get

$$\boxed{P = \frac{1}{2}\epsilon_0 \langle \vec{E}^2 \rangle + \frac{1}{2\mu_0}\langle \vec{B}^2 \rangle,} \tag{19}$$

so the pressure is equal to the (time averaged) energy density in the wave.

b) Near the earth, the energy per unit volume of sunlight is the solar constant C divided by the speed of light c, so that the pressure P is

$$P = C/c = 1.4 \times 10^3\ \mathrm{W/m^2} \div 3 \times 10^8 \mathrm{m/s} = 4.67 \times 10^{-6}\ \mathrm{N/m^2}. \tag{20}$$

For the density $\sigma = 10\ \mathrm{kg/m^2}$,

$$a = \frac{F}{m} = \frac{P}{\sigma} = 4.67 \times 10^{-7} \mathrm{m/s^2}. \tag{21}$$