Gaussian beams

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For a wave traveling in the z direction, a compact profile beam might be gaussian in the transverse direction. But most beams diffract and spread, so that the width of the gaussian would depend on how far the beam had propagated, meaning the width would be z-dependent. If the spread is not fast, the main z-dependence would still be the wave number oscillation. For the scalar case, or for a single component of a vector field, we try a solution

$$\Phi(\vec{x},t) = u(\vec{x})e^{i(kz-\omega t)} = u(x,y,z)e^{i(kz-\omega t)}, \tag{1}$$

with $k = \omega/c$, and use it to the Helmholtz equation supposing the derivative $\partial u/\partial z$ is small compared to ku, and that $\partial^2 u/\partial z^2$ is really small. This is the "paraxial" approximation.

The Helmholtz equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\vec{x},t) = 0. \tag{2}$$

With,

$$\begin{aligned}
\partial_z \Phi &= \left(iku + \frac{\partial u}{\partial z}\right)e^{i(kz-\omega t)}, \\
\partial_z^2 \Phi &= \left(-k^2 u + 2ik\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2}\right)e^{i(kz-\omega t)},
\end{aligned} \tag{3}$$

and dropping the second z-derivative of u, obtain

$$\left(\nabla_\perp^2 + 2ik\frac{\partial}{\partial z}\right)u(x,y,z) = 0. \tag{4}$$

In cylindrical coordinates, when there is azimuthal symmetry, this is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + 2ik\frac{\partial u}{\partial z} = 0. \tag{5}$$

For the trial solution, take a spreading gaussian,

$$u(\rho,z) = e^{i(f(z)+g(z)\rho^2)} \tag{6}$$

(where g can be a complex function, to get a compact gaussian). Then,

$$\begin{aligned}
\partial_z u &= i\left(f' + g'\rho^2\right)u, \\
\partial_\rho u &= 2ig\rho u, \\
\frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho u\right) &= 4igu + 2ig\rho\,\partial_\rho u = 4igu - 4g^2\rho^2 u.
\end{aligned} \tag{7}$$

Substituting,

$$4ig - 2kf' - \left(4g^2 + 2kg'\right)\rho^2 = 0. \tag{8}$$

This equation must be true at all ρ, so

$$g' = -\frac{2}{k}g^2. \tag{9}$$

This is a linear equation,

$$\frac{\partial}{\partial z}\left(\frac{1}{g(z)}\right)=\frac{2}{k}, \tag{10}$$

with solution

$$\begin{aligned} \frac{1}{g(z)} &= \frac{2}{k}\left(z-iz_R\right), \\ g(z) &= \frac{kz}{2(z^2+z_R^2)}+\frac{ikz_R}{2(z^2+z_R^2)}, \end{aligned} \tag{11}$$

where we chose the integration constant to be pure imaginary.
The solution so far is

$$u(\rho,z)=\exp\left[-\frac{kz_R\rho^2}{2(z^2+z_R^2)}\right]\exp\left[\frac{ikz\rho^2}{2(z^2+z_R^2)}\right]e^{if(z)}. \tag{12}$$

Also from Eq. (8),

$$\begin{aligned} f'(z) &= \frac{2i}{k}g(z)=\frac{i}{z-iz_R}, \\ f(z) &= i\int_0^z\frac{dz'}{z'-iz_R}=i\ln\left(z'-iz_R\right)\Big|_0^z=i\ln\left(1+i\frac{z}{z_R}\right). \end{aligned} \tag{13}$$

Since

$$1+i\frac{z}{z_R}=\left(\frac{z^2+z_R^2}{z_R^2}\right)^{1/2}e^{i\psi(z)}, \tag{14}$$

where $\psi(z)=\arctan\left(z/z_R\right)$, which is sometimes called the Guoy phase, the full solution is

$$\Phi(\vec{x},t)=u(\vec{x})e^{i(kz-\omega t)}=\left(\frac{z_R^2}{z^2+z_R^2}\right)^{1/2}\exp\left[-\frac{kz_R\rho^2}{2(z^2+z_R^2)}\right]\exp\left\{i\left[kz+\frac{kz\rho^2}{2(z^2+z_R^2)}-\psi(z)-\omega t\right]\right\}. \tag{15}$$

The minimum width of the gaussian is given by the minimum width parameter

$$w_0^2=\frac{2z_R}{k}, \tag{16}$$

which can allow us to determine z_R if some starting width is known,

$$z_R=\frac{1}{2}kw_0^2=\frac{\pi w_0^2}{\lambda}. \tag{17}$$

Thus z_R can get fairly large if w_0 is macroscopic and wavelength λ is for visible light. The width at general z is given in terms of the width function

$$w_z^2=\frac{2(z^2+z_R^2)}{kz_R}=w_0^2\left(1+\frac{z^2}{z_R^2}\right). \tag{18}$$

Turning to the phase term and parameterizing surfaces of constant phase as $(\rho,z)=(\rho,z_{\text{on-axis}}+\delta z)$, the first two terms in the phase (realizing that $k\gg 1/z_R$) give a surface of constant phase if

$$k\left(z+\frac{z\rho^2}{2(z^2+z_R^2)}\right)=\text{constant}\approx k\left(z_{\text{on-axis}}+\delta z+\frac{z\rho^2}{2(z^2+z_R^2)}\right), \tag{19}$$

neglecting the difference between z and $z_{\text{on-axis}}$ in the ρ^2 term. The relation between δz and ρ^2, for fixed $z_{\text{on-axis}}$, corresponds to a spherical wavefront of radius

$$R_z = \frac{z^2 + z_R^2}{z}. \tag{20}$$

The flat wavefront at $z = 0$ becomes curved as it propagates.

With the extra notation, the paraxial Helmholtz equation solution is

$$\boxed{\Phi(\vec{x}, t) = (w_0/w_z)\, e^{-\rho^2/w_z^2}\, e^{i(kz + k\rho^2/2R_z - \psi(z) - \omega t)}.} \tag{21}$$

Main source: Course notes from University of Colorado,

Instructor: C. Tim Lei, Course: Physics 4510 Optics

General webpage: `http://www.colorado.edu/physics/phys4510/phys4510_fa05/`

Specific webpage: `http://www.colorado.edu/physics/phys4510/phys4510_fa05/Chapter5.pdf`

Notes apparently from Fall 2005.

Note on opening angle: At long distances, $z \gg z_R$, the width parameter of the beam is

$$\lim_{z\to\infty} w(z) = z\,\frac{2}{kw_0} = z\,\frac{\lambda}{\pi w_0}, \tag{22}$$

and the opening angle of the beam (axis to edge) at long distances is

$$\Theta = \arctan\left(\frac{\lambda}{\pi w_0}\right). \tag{23}$$

Two examples,

$$\Theta = \begin{cases} 0.234\,\text{rad.} = 13.4^\circ & \lambda = 30\,\text{mm}, w_0 = 40\,\text{mm} \\ 0.21\,\text{mrad.} = 0.012^\circ & \lambda = .65\times10^{-3}\,\text{mm (red light)}, w_0 = 1\,\text{mm} \end{cases} \tag{24}$$

In the latter case, for $z = 55$ m (about 2 m less than the full length of the 1^{st} floor corridor in Small Physical Laboratory), the spot radius would be about 11.5 mm or 23 mm diameter. A measurement with an available red laser pointer gave about twice this. (Thanks to M. Andriotty for assistance.)

Selected figures (two from Mathematica notebook "GaussianBeam.nb" and one from Wikipedia)