Classical E&M II **Physics 611**

Final Exam | 3 hours | 10 or 16 December 2015, 9:00 am

Turn in this question sheet with answers, plus your work sheets. There are 3 problems.

Open text (Jackson only), open notes (this class only), open old homework (this class only).

Name: ______________________________

Three worksheets are attached.

1. [30 points total]
a) With the Born approximation, compute the unpolarized differential cross section for the scattering electromagnetic radiation (with wave number k) off a uniform dielectric sphere of radius a and relative dielectric constant $\epsilon_r = \epsilon/\epsilon_0 \approx 1$. [Note the word "unpolarized." Any polarization sums or averages must be done.]

b) Show that in the limit $ka \ll 1$ the differential cross section reduces to the small sphere result found in your notebooks, and which may be expressed as,

$$\frac{d\sigma}{d\Omega} = \frac{1}{18} k^4 a^6 \, |1 - \epsilon_r|^2 \, (1 + \cos^2\theta).$$

c) Express the $q = |\vec{q}|$ that may have appeared in part (a) in terms of k and the angle θ between the incoming and outgoing waves. Recall $\vec{q} = k(\hat{n}_0 - \hat{n})$ and where $\hat{n}_0$ is the direction of the incoming wave and $\hat{n}$ is the direction of the outgoing wave,

d) For $ka \gg 1$, choose the statement that is closest to true:
(i) The differential cross section is highly peaked in the forward direction, and in particular quite large at $\theta = 0$.
(ii) The differential cross section is highly peaked in the forward direction, although zero at exactly $\theta = 0$.
(iii) The differential cross section still has broad support over a wide range of angles.
(iv) None of the above is close to true.

Answers:

a)
$$\frac{d\sigma}{d\Omega} = \frac{1}{2} k^4 a^6 |\epsilon_r - 1|^2 \left(1 + \cos^2\theta\right) \left(\frac{j_1(qa)}{qa}\right)^2 \qquad \text{or equivalent}$$

b) Will find on work sheets.

c) $q = 2k \sin(\theta/2)$

d) Write a roman numeral here: (i).

2. [30 points total] Charges $+q$ and $-q$ exist at the ends of a neutral rod of length d. The rod rotates in the x-y plane with constant angular velocity ω about the z axis which passes through its center.

a) If we express the dipole moment as $\vec{p}(t) = \mathrm{Re}\,(\vec{p}e^{-i\omega t})$, find $\vec{p}$.

b) Compute the radiated power per unit solid angle $dP/d\Omega$ to leading non-trivial order (electric dipole radiation).

c) Find the total radiated power P.

Answers:

a) $\vec{p} = qd\,(1, i, 0)$

b)

$$\frac{dP}{d\Omega} = \frac{ck^4 q^2 d^2}{32\pi^2 \epsilon_0}\left(1 + \cos^2\theta\right)$$

c)

$$P = \frac{ck^4 q^2 d^2}{6\pi\epsilon_0}$$

3. [40 points total]
(a) A rectangular waveguide made from a nearly perfect conductor has sides of length a and b, in the x and y directions, respectively. Consider the TM modes ($E_z \neq 0$) and starting with an expression like

$$E_z(x, y) = (A\cos k_x x + B\sin k_x x)\left(C\cos k_y y + D\sin k_y y\right)$$

find an expression for $E_z(x, y)$ that satisfies the boundary conditions. (If you get help from a footnote in Jackson, you should work out why the information found there is correct.)

(b) Give an expression for $\gamma^2 = \epsilon_0\mu_0\omega^2 - k^2$ (where k is the wavenumber in the z-direction in the full expression $E_z(x, y, z, t) = \mathrm{Re}\,E_z(x, y)e^{i(kz-\omega t)}$). Also specifically give γ^2_{lowest} for the lowest nontrivial mode.

(c) Find the transverse electric field $\vec{E}_t$.

(d) Discuss (briefly) how you would proceed to calculate the energy loss in the case that the conducting walls of the waveguide were not perfect conductors.

Answers:

a) $E_z(x, y) = E_0 \sin\frac{\pi m x}{a} \sin\frac{\pi n y}{b}$,

b) $\gamma^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$,

$\gamma^2_{\text{lowest}} = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$,

c) $\vec{E}_t = \frac{i\pi k}{\gamma^2}\left(\frac{m\hat{x}}{a}\cos\frac{\pi m x}{a}\sin\frac{\pi n y}{b} + \frac{n\hat{y}}{b}\sin\frac{\pi m x}{a}\cos\frac{\pi n y}{b}\right)$.

d) Discussion: will find on work sheets.

1. One place to start is

$$\frac{\epsilon^* \cdot A_{sc}}{D_0} = \frac{k^2}{4\pi} \int d^3x\, e^{i\vec{q}\cdot\vec{x}} \left\{ \epsilon^* \cdot \epsilon_0 \frac{\delta\epsilon(x)}{\epsilon_0} + (n \times \epsilon^*) \cdot (n_0 \times \epsilon_0) \frac{\delta\mu(x)}{\mu_0} \right\} \tag{1}$$

with

$$\frac{d\sigma}{d\Omega} = \frac{|\epsilon^* \cdot A_{sc}|^2}{|D_0|^2}. \tag{2}$$

For the case at hand,

$$\begin{aligned}
\frac{\epsilon^* \cdot A_{sc}}{D_0} &= \frac{k^2}{4\pi} \epsilon^* \cdot \epsilon_0 (\epsilon_r - 1) \int_0^a r^2 dr \int_{-1}^1 2\pi d(\cos\theta)\, e^{iqr\cos\theta} \\
&= \frac{k^2(\epsilon_r - 1)}{2iq} \epsilon^* \cdot \epsilon_0 \int_0^a r dr \left(e^{iqr} - e^{-iqr} \right) \\
&= \frac{k^2(\epsilon_r - 1)}{2iq} \epsilon^* \cdot \epsilon_0 \frac{\partial}{i\partial q} \int_0^a dr \left(e^{iqr} + e^{-iqr} \right) = -\frac{k^2(\epsilon_r - 1)}{q} \epsilon^* \cdot \epsilon_0 \frac{\partial}{\partial q} \left(\frac{\sin qa}{q} \right) \\
&= k^2 a^3 (\epsilon_r - 1)\, \epsilon^* \cdot \epsilon_0\, \frac{\sin qa - qa\cos qa}{q^3 a^3}
\end{aligned} \tag{3}$$

If desired, one can also write this using a spherical Bessel function,

$$\frac{\epsilon^* \cdot A_{sc}}{D_0} = k^2 a^3 (\epsilon_r - 1)\, \epsilon^* \cdot \epsilon_0\, \frac{j_1(qa)}{qa}. \tag{4}$$

The differential cross section with all polarizations still indicated is

$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6 |\epsilon_r - 1|^2 |\epsilon^* \cdot \epsilon_0|^2 \left(\frac{j_1(qa)}{qa} \right)^2.} \tag{5}$$

There are four polarization possibilities. If the incoming and outgoing polarizations are both in the scattering plane, one gets $\epsilon^* \cdot \epsilon_0 = \cos\theta$. If they are both perpendicular to the scattering plane, $\epsilon^* \cdot \epsilon_0 = 1$. The other possibilities give zero. Hence with the average over initial polarizations and the sum over final,

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2} k^4 a^6 |\epsilon_r - 1|^2 \left(1 + \cos^2\theta\right) \left(\frac{j_1(qa)}{qa} \right)^2.} \tag{6}$$

(b) For $ka \to 0$ one also has $qa \to 0$, and we can use $j_1(qa) = (1/3)qa$ for small arguments (or the equivalent using the longer form farther above), so that

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{18} k^4 a^6 |\epsilon_r - 1|^2 \left(1 + \cos^2\theta\right).} \tag{7}$$

(c)

$$q^2 = (\vec{q})^2 = k^2(n^2 + n_0^2 - 2n \cdot n_0) = 2k^2(1 - \cos^2\theta) = 4k^2 \sin^2\frac{\theta}{2}, \tag{8}$$

$$\boxed{q = 2k\sin\frac{\theta}{2}.} \tag{9}$$

(d) For large ka, the cross section is forward peaked, and not zero at $\theta = 0$: (i). (The Bessel function $j_1(0) = 0$, but the limit $(j_1(qa)/qa)_{q\to 0}$ is finite. For larger angles, $j_1(qa)$ is limited in magnitude, and qa is rising seriously.)

2. (a) The charge $+q$ is located at

$$\vec{x} = \frac{d}{2}(\cos\omega t, \sin\omega t, 0)\,, \tag{10}$$

and the charge $-q$ is opposite this. Hence

$$\vec{p}(t) = \sum q_i \vec{x}_i(t) = qd\,(\cos\omega t, \sin\omega t, 0) = qd\,\mathrm{Re}\left(1, i, 0\right)e^{-i\omega t}\,, \tag{11}$$

and

$$\boxed{\vec{p} = qd\,\left(1, i, 0\right).} \tag{12}$$

(b) Electric dipole radiation radiates power in different directions according to

$$\frac{dP}{d\Omega} = \frac{ck^4}{32\pi^2\epsilon_0}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2. \tag{13}$$

Work out:

$$\begin{aligned} (\hat{n}\times\vec{p})\times\hat{n} &= \vec{p} - \hat{n}(\hat{n}\cdot\vec{p}) \\ |(\hat{n}\times\vec{p})\times\hat{n}|^2 &= |\vec{p}|^2 - (\hat{n}\cdot\vec{p})(\hat{n}\cdot\vec{p}^*) = q^2d^2\left(2-\sin^2\theta\right) = q^2d^2\left(1+\cos^2\theta\right) \end{aligned} \tag{14}$$

Hence,

$$\boxed{\frac{dP}{d\Omega} = \frac{ck^4q^2d^2}{32\pi^2\epsilon_0}\left(1+\cos^2\theta\right).} \tag{15}$$

(c) Integrate $(1+\cos^2\theta)$ over solid angle to get $(4/3)4\pi$, and

$$\boxed{P = \frac{ck^4q^2d^2}{6\pi\epsilon_0}.} \tag{16}$$

3. $E_z(x, y)$ for a waveguide gives the (x,y) dependence of $E_z(x, y, z, t) = E_z(x, y)e^{i(kz-\omega t)}$ and satisfies an equation

$$(\partial_x^2 + \partial_y^2 + \gamma^2)E_z(x, y) = 0, \tag{17}$$

and is solved by functions of the form,

$$E_z(x, y) = (A\cos k_x x + B\sin k_x x)\left(C\cos k_y y + D\sin k_y y\right). \tag{18}$$

(a) With the boundary conditions, namely that $E_z = 0$ at the sides of the waveguide, the solution becomes (in some suitable coordinate system),

$$\boxed{E_z(x, y) = E_0 \sin\frac{\pi m x}{a}\sin\frac{\pi n y}{b},} \tag{19}$$

where m and n are integers ≥ 1.

(b) From the solution and the defining equation,

$$\boxed{\gamma^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).} \tag{20}$$

The lowest mode comes when $m = n = 1$, leading to,

$$\boxed{\gamma^2_{\text{lowest}} = \pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right).} \tag{21}$$

(c) The x and y components of the electric field are found from

$$\vec{E}_t(x, y) = \frac{ik}{\gamma^2}\vec{\nabla}_t E_z(x, y), \tag{22}$$

so

$$\boxed{\vec{E}_t(x, y) = \frac{i\pi k}{\gamma^2}\left(\frac{m\hat{x}}{a}\cos\frac{\pi m x}{a}\sin\frac{\pi n y}{b} + \frac{n\hat{y}}{b}\sin\frac{\pi m x}{a}\cos\frac{\pi n y}{b}\right).} \tag{23}$$

(d) The energy loss in the conductor can be obtained from the surface resistance and the tangential magnetic field at the surface of the conductor. The energy loss per unit area of conductor surface is

$$\frac{dP_{\text{loss}}}{dA} = \rho_{\text{surf}}\,|\vec{K}|^2 = \sqrt{\frac{\mu_c\omega}{2\sigma}}\,|\vec{H}_{\text{tang}}|^2, \tag{24}$$

where ρ_{surf} is the surface resistance and $\vec{K}$ is the surface current, whose magnitude is numerically equal to the magnitude of the component of $\vec{H}$ tangential to the surface under study. We would need to know the conductivity of the conductor, σ, have to be told the frequency ω, be told the permeability of the conductor μ_c, which could be close to that of free space but isn't always. We would also have to work out the x and y components of the magnetic field, which can be done from

$$\vec{H}_t(x, y) = \frac{i\epsilon\mu\omega}{\gamma^2}\hat{z}\times\vec{\nabla}_t E_z(x, y), \tag{25}$$

and figure out which components were tangential to the surface. (For information, the "t" in H_t stands for transverse—to the long direction of the waveguide—not tangential.)