Forces on small, neutral objects.

Can also calculate more directly, and we'll find
Force on small neutral dielectric ~~particles~~ object from oscillating EM fields.
(sub. for 6.8, 6.9 in Jackson) ← monochromatic

One particle: $F = qE + q\vec{v} \times \vec{B}$

Collection of point charges:

$$F = \sum q_i \vec{E}(x_i, t) + \sum q_i \dot{\vec{x}}_i \times \vec{B}(x_i, t)$$

Case: oscillating fields vary over length scale λ.
say size of ~~particles~~ object $\ll \lambda$ [$\langle r \rangle \ll \lambda$]

Expand:

$$E(x_i, t) = E(x_0, t) + ((\vec{x}_i - \vec{x}_0) \cdot \vec{\nabla}) E(x_0, t) + \ldots$$

$$B(x_i, t) = B(x_0, t) + \ldots \quad \text{(enough)}$$

$$F = \left(\sum q_i\right) E(x_0, t) + \left(\sum q_i (\bar{x}_i - \bar{x}_0) \cdot \vec{\nabla}\right) E(x_0, t) + \ldots$$
$$+ \sum q_i \, (\dot{\vec{x}}_i - \dot{\vec{x}}_0) \times \vec{B}(x_0, t)$$

← freebie

Neutral:

$$\sum q_i = 0$$
$$\sum q_i (\bar{x}_i - \bar{x}_0) \equiv \bar{p} \quad \text{dipole moment.}$$

$$\boxed{F = (\vec{p} \cdot \vec{\nabla}) E(x_0, t) + \dot{\vec{p}} \times \vec{B}(x_0, t)}$$

plus corr. of $\mathcal{O}(\langle r \rangle / \lambda)$

L4
9/8/2015

Dielectric: polarization driven by electric field.
take linear
static case: $\vec{p} = \alpha \vec{E}$.

Here oscillating : $\vec{E}(x,t) = E_0(x)\cos(\omega t - \varphi)$

because of inertia may be time lag,

$\vec{p}(x,t) = \vec{p}_0(x)\cos(\omega(t - t_{lag}) - \varphi)$

$= \vec{p}_0(x)\cos(\omega t - \varphi')$

~~Useful~~

Will use complex notation: e.g., $\vec{E}(x,t) = \mathrm{Re}\ E_0(x)e^{-i\omega t}$

E_0 may be complex (e.g., $E_0(x) = E(x)e^{i\varphi}$)

$\vec{p}(x,t) = \mathrm{Re}\ p_0(x)e^{-i\omega t}$

$p_0 = \alpha E_0$ α may be complex, to account for phase difference.

Will consider time averaged force

$\langle \ldots \rangle$ for time average.

Note that $\langle A \cdot B \rangle = \frac{1}{2}\mathrm{Re}(A_0 \cdot B_0^*)$ (where A, B both oscillate as above, as same ω)

~~Proof~~

Proof:

$$A = \mathrm{Re}\ A_0 e^{-i\omega t}$$
$$= \mathrm{Re}\left[(\mathrm{Re}A_0 + i\,\mathrm{Im}A_0)(\cos\omega t - i\sin\omega t)\right]$$
$$= \mathrm{Re}\left[\mathrm{Re}A_0\cos\omega t + \mathrm{Im}A_0\sin\omega t + 2\text{ more terms}\right]$$
$$= \mathrm{Re}A_0\cos\omega t + \mathrm{Im}A_0\sin\omega t$$

$$\langle A \cdot B \rangle = \mathrm{Re}A_0 \cdot \mathrm{Re}B_0\langle\cos^2\omega t\rangle + \mathrm{Im}A_0 \cdot \mathrm{Im}B_0\langle\sin^2\omega t\rangle + \mathrm{Re}A_0 \cdot \mathrm{Im}B_0\langle\cos\omega t\sin\omega t\rangle + \mathrm{Im}A_0\,\mathrm{Re}B_0 \ldots$$

$$= \frac{1}{2}(\mathrm{Re}A_0 \cdot \mathrm{Re}B_0 + \mathrm{Im}A_0 \cdot \mathrm{Im}B_0)$$

$$= \frac{1}{2}\mathrm{Re}(A_0 \cdot B_0^*)$$

QED

(just in case: $A_0 \cdot B_0^* = (\mathrm{Re}\,A_0 + i\,\mathrm{Im}\,A_0)(\mathrm{Re}\,B_0 - i\,\mathrm{Im}\,B_0)$

$$= \mathrm{Re}\,A_0\,\mathrm{Re}\,B_0 + \mathrm{Im}\,A_0\,\mathrm{Im}\,B_0 + i\,\mathrm{Im}\,A_0\,\mathrm{Re}\,B_0 - i\,\mathrm{Re}\,A_0\,\mathrm{Im}\,B_0 .$$)

Time derivative:

$$p(x,t) = \mathrm{Re}\left(p_0(x)e^{-i\omega t}\right)$$
$$\dot{p}(x,t) = \mathrm{Re}\left(-i\omega\, p_0(x)e^{-i\omega t}\right)$$

(again just in case: really works. $p_0 = |p_0|e^{i\varphi'}$

$$p(x,t) = \mathrm{Re}\ |p_0|e^{-i(\omega t-\varphi')} = |p_0|\cos(\omega t-\varphi')$$

$$\dot{p} \stackrel{?}{=} \mathrm{Re}\left(-i\omega\, p_0 e^{-i\omega t}\right) = \mathrm{Re}\left(-i\omega\, |p_0| e^{-i(\omega t-\varphi')}\right)$$
$$= \mathrm{Re}\left[-i\omega\,|p_0|\left(\cos(\omega t-\varphi') - i\sin(\omega t-\varphi')\right)\right]$$
$$= -\omega\,|p_0|\sin(\omega t-\varphi'). \quad \checkmark$$)

Time averaged force

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})E + \dot{\vec{p}}\times\vec{B}$$

$$\langle F\rangle = \frac{1}{2}\mathrm{Re}\left((p_0\cdot\nabla)E_0^*\right) + \frac{1}{2}\mathrm{Re}\left(-i\omega\vec{p}_0\times\vec{B}_0^*\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(\alpha\,(E_0\cdot\nabla)E_0^*\right) + \frac{1}{2}\mathrm{Re}\left(-i\alpha\omega\,E_0\times B_0^*\right)$$

Faraday:

$$\nabla\times E = -\frac{\partial B}{\partial t}$$
$$\therefore\quad \nabla\times E_0 = +i\omega B_0$$
$$\nabla\times E_0^* = -i\omega B_0^*$$

$$\langle F\rangle = \frac{1}{2}\mathrm{Re}\left(\alpha(E_0\cdot\nabla)E_0^* + \alpha E_0\times(\nabla\times E_0^*)\right)$$

use

$$E_0\times(\nabla\times E_0^*) = \vec{E}_0\cdot(\vec{\nabla})\vec{E}_0^* - (\vec{E}_0\cdot\vec{\nabla})\vec{E}_0^*$$

$$\therefore\quad \boxed{\langle F\rangle = \frac{1}{2}\mathrm{Re}\left[\alpha\,\vec{E}_0\cdot(\vec{\nabla})\vec{E}_0^*\right]}$$

Further analyze: α may be complex.

$$\alpha = \alpha' + i\alpha'', \quad \alpha', \alpha'' \text{ real.}$$

$$\langle F\rangle = \frac{1}{2}\,\mathrm{Re}\left[(\alpha'+i\alpha'')\,\vec{E}_0\cdot(\vec{\nabla})E_0^*\right]$$
$$= \frac{1}{2}\alpha'\,\mathrm{Re}\left[\vec{E}_0\cdot(\vec{\nabla})\vec{E}_0^*\right] - \frac{1}{2}\alpha''\,\mathrm{Im}\left[\vec{E}_0\cdot(\nabla)\vec{E}_0^*\right]$$

$$\langle F\rangle_{\alpha'} = \frac{1}{2}\alpha'\cdot\frac{1}{2}\left\{E_0\cdot(\vec{\nabla})E_0^* + E_0^*\cdot(\vec{\nabla})E_0\right\}$$
$$= \frac{1}{4}\alpha'\,\vec{\nabla}(E_0\cdot E_0^*)$$
$$= \frac{1}{4}\alpha'\,\vec{\nabla}|\vec{E}_0|^2$$

The intensity gradient force – comment any later.

$\langle F\rangle_{\alpha''}$: first consider momentum density in fields from Poynting's theorem,

$$\vec{g} = \varepsilon_0\,\vec{E}\times\vec{B} \qquad = \frac{1}{c^2}\vec{S}$$
$$\langle\vec{g}\rangle = \frac{1}{2}\varepsilon_0\,\mathrm{Re}\,(E_0\times B_0^*)$$
$$B_0^* = \frac{i}{\omega}\nabla\times E_0^*$$

$$\langle\bar{g}\rangle = \frac{\varepsilon_0}{2\omega}\,\mathrm{Re}\left\{iE_0\times(\nabla\times E_0^*)\right\}$$
$$= -\frac{\varepsilon_0}{2\omega}\,\mathrm{Im}\left\{E_0\times(\nabla\times E_0^*)\right\}$$
$$= -\frac{\varepsilon_0}{2\omega}\,\mathrm{Im}\left\{E_0\cdot(\vec{\nabla})E_0^* - (E_0\cdot\nabla)E_0^*\right\}$$

Next consider $\vec{\nabla}\times(E_0\times E_0^*) =$ 4 terms, two where ∇ acts on E_0, and two where it acts on E_0^*.

$$= \bar{E}_0(\nabla\cdot E_0^*) - (E_0\cdot\nabla)E_0^* - E_0^*(\nabla\cdot E_0) + (E_0^*\cdot\nabla)E_0$$
$$= 2i\,\mathrm{Im}\left[E_0(\nabla\cdot E_0^*)\right] - 2i\,\mathrm{Im}\left[(E_0\cdot\nabla)E_0^*\right]$$

For this term, use $\nabla\cdot E_0 = 0$. Meaning we make approx. for this term by just keeping incoming wave (for which we can consider $\rho = 0$).

Then

$$\langle\vec{g}\rangle = -\frac{\varepsilon_0}{2\omega}\,\mathrm{Im}\left[E_0\cdot(\vec{\nabla})E_0^*\right] + \frac{i\varepsilon_0}{4\omega}\,\cancel{\mathrm{Im}}\left[\nabla\times(E_0\times E_0^*)\right]$$

$$\frac{1}{2}\,\mathrm{Im}\left[E_0\cdot(\vec{\nabla})E_0^*\right] = -\frac{2\omega}{\varepsilon_0}\langle \vec{g}\rangle + \frac{i}{2}\vec{\nabla}\times(E_0\times E_0^*)$$

~~Define $\vec{s}(r)$ for here~~ lower case

$$\therefore\ \langle F\rangle = \frac{1}{4}\alpha'\vec{\nabla}|E_0|^2 + \frac{\omega\alpha''}{\varepsilon_0 c^2}\langle \vec{S}\rangle - \frac{i}{4}\vec{\nabla}\times(E_0\times E_0^*)$$

Albaladejo et al.

Note that last term is real.

Often define $\langle \vec{L}_s\rangle = \frac{\varepsilon_0}{2\omega}\, i\,(\bar{E}_0\times \bar{E}_0^*)$, $\frac{\varepsilon_0}{4\omega i}$

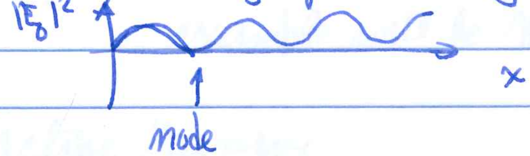
whereupon

$$\boxed{\langle \vec{F}\rangle = \frac{1}{4}\alpha'\vec{\nabla}|E_0|^2 + \frac{\omega\alpha''}{\varepsilon_0 c^2}\langle \vec{S}\rangle + \frac{\omega\alpha''}{\varepsilon_0}\vec{\nabla}\times\langle \vec{L}_s\rangle}$$

- for plane waves, 1^{st} and 3^{rd} terms are zero
  get result that force proportional to Poynting vector.

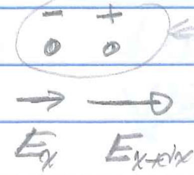
- still regarding plane waves, $E_0\times E_0^* = 0$ for linear polarization
  but $E_0\times E_0^* \neq 0$ for circular polarization.
  but for plane waves, $E_0\times E_0^*$ has no position dep.

- example where first term not zero, think of two opposing lasers setting up standing wave.



Force pushes objects to max $|E_0|^2$
"optical tweezer"

- more physically, $\vec{E}$ polarizes object,
  if $\frac{\partial E}{\partial x}\neq 0$, push to stronger E field:



net force to right.

- Last term is spin curl force. Requires weird but not unthinkable field configurations.

Refs: Chaumet & Nieto-Vesperinas, Opt. Lett. 25 (2000), 1065-1067
Albaladejo, Marqués, Laroche, and Sáenz, PRL 102 (2009), 113602 (4pp)

see also

Ruffner and Grier, PRL 111 (2013), 059301 (1 pp),
and response by Marqués and Sáenz, PRL 111 (2013), 059302 (1 pp).

For those who might like to see a case where the Poynting vector points one way but the radiation pressure pushes another way may like:

Afanasev, Carlson, and Mukherjee, J. Opt. Soc. Am. B 31 (2014), 2721-2727.