Quantum Field Theory **Physics 722**

Final Exam | Three hours | 11 May 2010

1. [25 points total] Let ψ be a Dirac field, which we call the electron field, let ϕ be a neutral scalar field, and let V^μ be a neutral vector field. Consider the following Lagrangians,
a) $\mathscr{L} = \mathscr{L}_0 - e\bar\psi\gamma_\mu\psi\, V^\mu$,
b) $\mathscr{L} = \mathscr{L}_0 - i f\,\bar\psi\gamma_5\psi\,\phi$,
c) $\mathscr{L} = \mathscr{L}_0 - h\bar\psi\psi\,\phi^2$,
where e, f, and h are constants and $\mathscr{L}_0$ is the free Lagrangian. For each Lagrangian, draw all the lowest nontrivial order Feynman diagrams that contribute to electron-positron scattering. (I believe there should be two diagrams in each case.) Using the Feynman diagrams as guidance, successfully write down the electron-positron scattering amplitudes for each case.

2. [35 points total] Consider a hypothetical decay $\Lambda^0 \to p + \pi^-$. For this problem, Λ^0 is a polarized spin-1/2 particle of mass m_Λ and momentum P; p is a massless (this is where the "hypothetical" comes in) spin-1/2 proton of momentum p; and π^- is a massless spin-0 particle of momentum k. Let the Lagrangian density be

$$\mathscr{L} = \bar\psi_p\left(g_V\gamma_\mu - g_A\gamma_\mu\gamma_5\right)\psi_\Lambda \times i\partial^\mu\phi + \text{hermitian conjugate}\,,$$

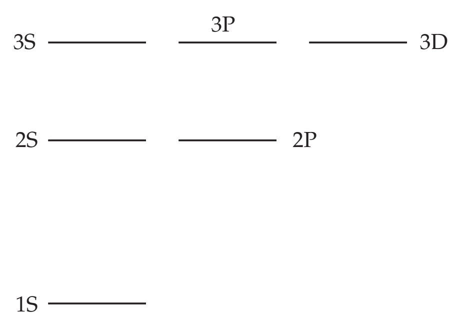
where g_V and g_A are real constants, and ϕ is the field for the π^-.
(a) Draw the Feynman diagram for the decay.
(b) Write down the amplitude $\mathscr{M}$ for the decay.
(c) Work the amplitude into the form

$$\mathscr{M} \propto m_\Lambda\, \bar u(p)\left(g_V + g_A\gamma_5\right)u(P)\,,$$

where the polarization arguments are tacit.
(d) Sum $|\mathscr{M}|^2$ over the appropriate polarizations.
Query: Did you have the minus sign in $\{\bar u(p)\left(g_V + g_A\gamma_5\right)u(P,S)\}^\dagger = \bar u(P,S)\left(g_V - g_A\gamma_5\right)u(p)$?
Further small note: The case of real Λ^0, real p, and real π^- is not so far from the above with $g_V \approx g_A$.

3. [20 points total] Evaluate the integral $\int d\alpha\, d\beta\, e^{\alpha M\beta}$ for the case that α and β are independent Grassmann variables, and M is an ordinary number (or a 1×1 matrix).

4. [20 points total] (a) For the three lowest energy levels of hydrogen, indicate the possible E1 and M1 radiative decays ($nL \to n'L' + \gamma$), labeling each as E1 or M1 as the case may be. To keep the diagram simpler, only draw decays where n and n' differ by one unit. Neglect energy differences among states with the same n.



(b) What do you expect is the main decay of the 3S state? Why?
(c) Of all the states in the diagram, which, if any, is (are) metastable? Why?

1: Feynman diagrams and scattering amplitudes for electron-positron scattering with
a) $\mathscr{L} = \mathscr{L}_0 - e\bar{\psi}\gamma_\mu\psi V^\mu$,
b) $\mathscr{L} = \mathscr{L}_0 - if\bar{\psi}\gamma_5\psi\phi$,
c) $\mathscr{L} = \mathscr{L}_0 - h\bar{\psi}\psi\phi^2$,

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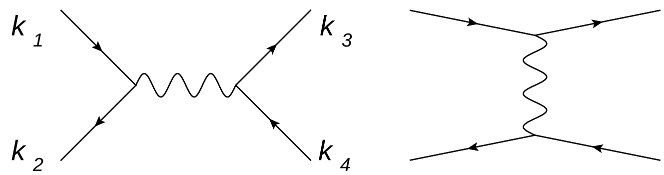
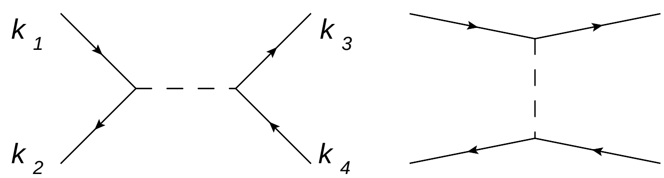
a)



$$i\mathscr{M} = (-ie)^2(-i)\left\{\frac{\bar{v}(k_2)\gamma^\mu u(k_1)\ \bar{u}(k_3)\gamma_\mu v(k_4)}{(k_1+k_2)^2} - \frac{\bar{u}(k_3)\gamma^\mu u(k_1)\ \bar{v}(k_2)\gamma_\mu v(k_4)}{(k_1-k_3)^2}\right\}$$

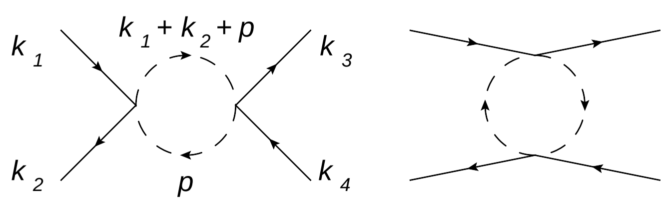
Note: the "–" sign comes because one pair (or an odd number of pairs) of fermion fields are interchanged when evaluating the second Feynman diagram

b)



$$i\mathscr{M} = (f)^2(i)\left\{\frac{\bar{v}(k_2)\gamma_5 u(k_1)\ \bar{u}(k_3)\gamma_5 v(k_4)}{(k_1+k_2)^2} - \frac{\bar{u}(k_3)\gamma_5 u(k_1)\ \bar{v}(k_2)\gamma_5 v(k_4)}{(k_1-k_3)^2}\right\}$$

c)



$$i\mathscr{M} = (-ih)^2(i^2)\int\frac{d^4p}{(2\pi)^4}\left\{\frac{\bar{v}(k_2)u(k_1)\ \ \bar{u}(k_3)v(k_4)}{p^2(k_1+k_2+p)^2} - \frac{\bar{u}(k_3)u(k_1)\ \ \bar{v}(k_2)v(k_4)}{p^2(k_1-k_3+p)^2}\right\}$$

SOLUTION FOR #2

2: $\Lambda^0 \to p + \pi^-$ w/ massless proton and $\mathscr{L} = \bar{\psi}_p \left(g_V \gamma_\mu - g_A \gamma_\mu \gamma_5\right) \psi_\Lambda \times i\partial^\mu \phi + \text{h. c.}$
(a) Draw the Feynman diagram for the decay.
(b) Write down the amplitude $\mathscr{M}$ for the decay.
(c) Work the amplitude into the form $\mathscr{M} \propto m_\Lambda\ \bar{u}(p) \left(g_V + g_A \gamma_5\right) u(P)$.
(d) Sum $|\mathscr{M}|^2$ over the appropriate polarizations.

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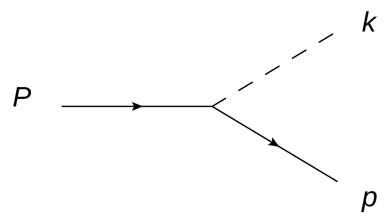
a)



b)

$$\mathscr{M} = \bar{u}(p) \left(g_V \gamma_\mu - g_A \gamma_\mu \gamma_5\right) u(P)\ k^\mu$$

c) Use $k = P - p$ to obtain

$$\begin{aligned}\mathscr{M} &= \bar{u}(p) \left(g_V (\not{P} - \not{p}) - g_A (\not{P} - \not{p}) \gamma_5\right) u(P) \\ &= \bar{u}(p) \left(g_V\ \not{P} + g_A \gamma_5\ \not{P}\right) u(P) \\ &= m_\Lambda \bar{u}(p) \left(g_V + g_A \gamma_5\right) u(P)\end{aligned}$$

d) Calculate $|\mathscr{M}|^2$ summed over Lambda and proton polarizations. (For use in decay rate calculation would divide by 2 for average over initial polarization.)

$$\begin{aligned}\sum_{pol} |\mathscr{M}|^2 &= m_\Lambda^2\ \text{Tr}\ \not{p} \left(g_V + g_A \gamma_5\right) (\not{P} + m_\Lambda) \left(g_V - g_A \gamma_5\right) \\ &= m_\Lambda^2\ \text{Tr}\ \not{p} \left(g_V + g_A \gamma_5\right) \not{P} \left(g_V - g_A \gamma_5\right) \\ &= m_\Lambda^2 \left\{g_V^2 \text{Tr}\ \not{p}\, \not{P} + g_A^2 \text{Tr}\ \not{p}\, \not{P}\right\} \\ &= 4 m_\Lambda^2 P \cdot p \left(g_V^2 + g_A^2\right)\end{aligned}$$

Professional touch:

$$\begin{aligned}(P - p)^2 &= k^2 \\ m_\Lambda^2 - 2P \cdot p + 0 &= 0 \\ 2P \cdot p &= m_\Lambda^2\end{aligned}$$

So:

$$\sum_{pol} |\mathscr{M}|^2 = 2 m_\Lambda^4 \left(g_V^2 + g_A^2\right)$$

3: Integrate $\int d\alpha\, d\beta\, e^{\alpha M \beta}$ where α and β are Grassmann.

Rules:

$$\int d\alpha = 0$$

$$\int d\alpha\ \alpha = 1$$

$$\alpha^2 = 0; \qquad \alpha\beta = -\beta\alpha$$

Then

$$\begin{aligned}\int d\alpha\, d\beta\ e^{\alpha M \beta} &= \int d\alpha\, d\beta\ \left(1 + \alpha M \beta\right) \\ &= 0 - M \int d\alpha \left(\int d\beta\ \beta\right) \alpha \\ &= -M\end{aligned}$$

4: (a) Draw E1 and M1 decays on diagram
(b) main decay of 3S
(c) metastable state(s)

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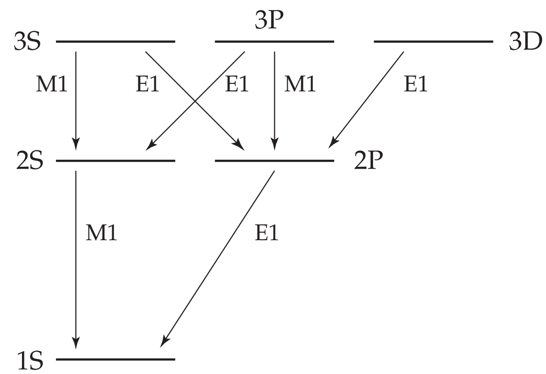
a)



b) The main decay of the 3S is the E1 transition to the 2P state.

Up to a simple numerical factor, the E1 and M1 decay rates go like

$$\Gamma(E1) \propto \alpha\omega^3 r_{fi}^2 \quad \text{and} \quad \Gamma(M1) \propto \alpha\omega^3 \frac{1}{m^2} \times \text{overlap integral}\,.$$

The size scale of the atom is given by the Bohr radius, so

$$r_{fi} \propto a_B = \frac{1}{m\alpha}$$

and the E1 rates are of order $(1/\alpha)^2 \approx 137^2$ bigger than the M1 rates, even without considering the overlap of the initial and final spatial wave functions.

c) The 2S state is metastable. The M1 decay is slow, and all the other states have at least one allowed E1 decay, which gives a quick decay.