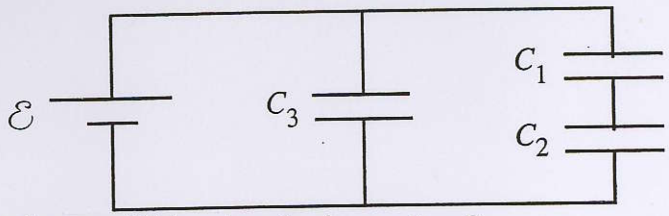
Problem 1.

Three capacitors, $C_1 = 2\ \mu F$, $C_2 = 2\ \mu F$, and $C_3 = 4\ \mu F$, are connected to a battery with voltage $\mathcal{E} = 10\text{V}$ as shown.



a) What is the equivalent capacitance of this network of capacitors?

b) What are the charges stored on each of the capacitors?

c) How much electrical energy is stored in C_3?

d) If C_3 is a parallel-plate capacitor with plates of area 0.5 mm^2, which is filled with a dielectric material with a dielectric constant of 5, how far apart are these plates?

a) C_1 & C_2 are in series $\therefore \frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\mu F} + \frac{1}{2\mu F} \Rightarrow C_s = 1\mu F$

C_3 is in parallel with this combination:

$\therefore C_{eq} = C_3 + C_s = 4\mu F + 1\mu F = 5\mu F$ $\boxed{C_{eq} = 5\mu F}$

b) $Q_3 = C_3 \mathcal{E} = (4\mu F)(10V) = 40\mu C$

$Q_1 = C_s \mathcal{E} = Q_2 = (1\mu F)(10V) = 10\mu C$

or since $C_1 = C_2$ we have $V_1 = V_2 = \frac{\mathcal{E}}{2}$ $\therefore Q_1 = C_1 V_1 = (2\mu F)\left(\frac{10V}{2}\right) = 10\mu C$

etc. for Q_2

$\boxed{Q_1 = Q_2 = 10\mu C \quad Q_3 = 40\mu C}$

c) potential energy $U_3 = \frac{1}{2} C_3 \mathcal{E}^2 = \frac{1}{2}(4\mu F)(10V)^2 = 200\mu J$

$\boxed{U_3 = 2 \times 10^{-4} J}$

d) $C = K\frac{\varepsilon_0 A}{d}$ $\therefore d = \frac{K\varepsilon_0 A}{C}$

$$= \frac{5\left(8.85\times10^{-12}\frac{C^2}{Nm^2}\right)\left(0.5\times10^{-6}m^2\right)}{4\times10^{-6}F}$$

$$= \boxed{5.5\times10^{-12} m} \quad (!??)$$

which is not physically possible (less than $\approx$ atomic radii) – I had a typo in the area, which was supposed to be much larger... sadly, nobody seems to have questioned this number!

Problem 2.

An electron (mass = 9.11×10^{-31} kg) is located in a region of space where the electric potential is given by $V(x) = (3.0\ \frac{\text{V}}{\text{m}^3})x^3 - (4.0\frac{\text{V}}{\text{m}})x$.

a) When the electron is located at $x = 2$ m, what is its acceleration vector?

b) How much work must be done by an external force to move the electron from $x = 2$ m to $x = 0$ m?

a) $V = 3x^3 - 4x$ (leaving off the units for the moment to simplify notation)

$$E_x = -\frac{dV}{dx} = -9x^2 + 4$$

$\therefore E_x(x=2) = -9(2)^2 + 4 = -32 \quad \therefore E_x(x=2\text{m}) = -32\ \text{V/m}$

$F = qE = ma \quad \therefore a = \frac{qE}{m} = \frac{(-1.6\times10^{-19}\text{C})(-32\ \text{V/m})}{9.11\times10^{-31}\ \text{Kg}} = 5.62\times10^{12}\ \text{m/s}^2$ (Wow!)

direction of $\vec{a}$ is $+x$ direction

$\therefore \boxed{\vec{a} = +5.62\times10^{12}\ \frac{\text{m}}{\text{s}}\ \hat{i}}$

b) Work done by external force $= \Delta U = q\Delta V$

$V(x=2\text{m}) = 3(2)^3 - 4(2) = 16\text{V}$

$V(x=0\text{m}) = 3(0)^3 - 4(0) = 0\text{V}$

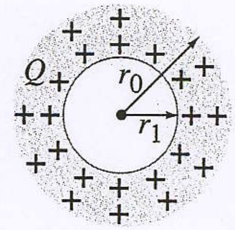
$\therefore \Delta V = -16\text{V}$

$\text{Work} = q\Delta V = (-1.6\times10^{-19}\text{C})(-16\text{V}) = \boxed{+2.56\times10^{-18}\text{J}}$

or $\boxed{+16\text{eV}}$

(from definition of the unit "eV")

Problem 3.



Consider a nonconducting sphere of radius r_0, which has a spherical cavity of radius r_1, centered at the sphere's center. An electric charge Q is distributed uniformly throughout the "shell" (*i.e.* between $r = r_1$ and $r = r_0$). Determine the electric field (magnitude and direction) as a function of r for:

a) $0 < r < r_1$

b) $r_1 < r < r_0$

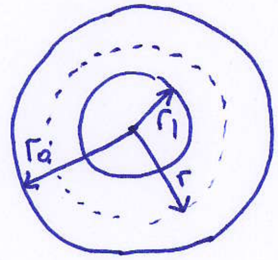
c) $r > r_0$

Clearly, Gauss' Law is applicable : $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$

a) $0<r<r_1$: choose spherical Gaussian surface of radius $r<r_1$
By symmetry, $\vec{E}$ is same everywhere on that surface
$Q_{encl} = 0$ $\quad \therefore \oint \vec{E}\cdot d\vec{A} = E\int dA = EA = \frac{0}{\varepsilon_0}$ $\quad \therefore \boxed{E = 0}$

b) $r_1<r<r_0$: again, spherical Gaussian surface of radius r ;
again by symmetry $\vec{E}$ is $\perp$ to surface & same magnitude everywhere on surface

$\therefore \oint \vec{E}\cdot d\vec{A} = \int E\,dA = EA = E4\pi r^2$



$Q_{enclosed} = Q * \text{fraction of shell's volume enclosed}$

$= Q\,\frac{V_{enclosed}}{V_{shell}}$

hollow cavity volume

$= Q\left(\frac{4/3\,\pi r^3 - 4/3\,\pi r_1^3}{4/3\,\pi r_0^3 - 4/3\,\pi r_1^3}\right)$ $\quad \therefore \vec{E} = \frac{1}{4\pi r^2}\frac{Q_{encl}}{\varepsilon_0}\hat{r}$

$$\boxed{\vec{E} = \frac{Q}{4\pi r^2}\left(\frac{r^3 - r_1^3}{r_0^3 - r_1^3}\right)\hat{r}}$$

c) $r>r_0$: another spherical Gaussian surface, this time with $r>r_0$
Once again, symmetry gives $\oint \vec{E}\cdot d\vec{A} = EA = E4\pi r^2$
$Q_{enclosed}$ here $= Q$

$\therefore E4\pi r^2 = \frac{Q}{\varepsilon_0}$ $\quad \therefore \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}}$

(same as field due to point charge Q located @ origin)

Problem 4.

You toast your morning bagel at the Cafeteria. Being a physics nerd, you calculate that 26.4 kJ of heat were produced by the toaster in one minute. The toaster is supplied by AC electricity at 110 V (rms) and 60 Hz, and it consists of, in essence, a single cylindrical nichrome ($\rho = 1 \times 10^{-6}\ \Omega \cdot \mathrm{m}$) wire of total length 3.0 m (the wire is coiled up). Ignore any change of resistivity with temperature.

a) What rms current runs through the wire when the toaster is on?

b) What is the peak current?

c) What is the resistance of the wire?

d) What is the diameter of the wire?

e) What is the drift velocity of the electrons in the wire, given that there are roughly 10^{28} free electrons/m^3 in nichrome?

a) Heat (energy) = power * time = $Pt = Q$

$\therefore\ P = \frac{Q}{t} = \frac{26.4 \times 10^3\,\mathrm{J}}{60\,\mathrm{s}} = 440\,\mathrm{W}$

note: one minute is many 60 Hz cycles $\therefore$ power is average power here.

$P = I_{rms} V_{rms}$ $\quad \therefore\ I_{rms} = \frac{P}{V_{rms}} = \frac{440\,\mathrm{W}}{110\,\mathrm{V}} = \boxed{4\,\mathrm{A}}$

b) $I_0 = \sqrt{2}\, I_{rms} = \boxed{5.66\,\mathrm{A}}$

c) $P = I_{rms}^2 R$ $\quad \therefore\ R = \frac{P}{I_{rms}^2} = \frac{440\,\mathrm{W}}{(4\,\mathrm{A})^2} = \boxed{27.5\,\Omega}$

d) $R = \frac{\rho l}{A}$ $\quad A = \pi r^2 = \frac{\pi d^2}{4}$

$\therefore\ d = \left(\frac{4\rho l}{\pi R}\right)^{1/2} = \left[\frac{4(10^{-6}\,\Omega\mathrm{m})(3\,\mathrm{m})}{\pi(27.5\,\Omega)}\right]^{1/2} = 3.73 \times 10^{-4}\,\mathrm{m}$

$= \boxed{0.373\,\mathrm{mm}}$

e) $I_0 = neAv_d$

note: use peak not rms current here

$\therefore\ v_d = \frac{I}{neA} = \frac{I}{ne\left(\frac{\pi d^2}{4}\right)} = \frac{5.66\,\mathrm{A}}{(10^{28}\,\mathrm{m}^{-3})(1.6 \times 10^{-19}\,\mathrm{C})\left[\frac{\pi(3.73 \times 10^{-4}\,\mathrm{m})^2}{4}\right]}$

$= \boxed{0.033\,\mathrm{m/s}}$

recall: electron drift velocities in ordinary materials are surprisingly slow...