Ground rules: you can use your notes, the textbook, mathematical tables, programs like Maple, MathCAD, Mathematica, etc. but not other quantum mechanics textbooks. No discussions with your colleagues.

1. [20 points] Consider a particle in a 1 dimensional “box”, i.e. where

\[ V(x) = 0 \quad \text{for } 0 < x < a \]

and

\[ V(x) = \infty \quad \text{elsewhere}. \]

Calculate the uncertainty product \( \Delta x \Delta p \) for the \( n \)th eigenstate, and compare to the Heisenberg uncertainty principle. Comment.

2. [15 points] For the 1-D harmonic oscillator, evaluate \( \langle x^4 \rangle \) for the \( n \)th energy eigenstate.


4. Consider a particle of mass \( m \) moving along the \( x \)-axis in the potential \( V(x) \) where

\[ V(x) = +\infty \quad \text{for } x < 0 \]

and

\[ V(x) = \frac{1}{2}m\omega^2 x^2 \quad \text{for } x > 0 \]

a) [10 points] Use the variational principle with the trial wavefunction \( \psi(x) = 2\lambda^{3/2}xe^{-\lambda x} \) (for \( x < 0 \)) and \( \psi(x) = 0 \) (for \( x > 0 \)) to estimate the ground state energy.

b) [5 points] What are the exact stationary-state energy spectrum and wavefunctions?

5. The deuteron (the nucleus of deuterium) is a bound state of a neutron and a proton. A useful approximation for the neutron-proton interaction is the potential

\[ V(r) = -Ae^{-r/a} \]

where \( A = 32 \) MeV, \( a = 2.2 \) fm, and \( r \) is the neutron-proton separation.

a) [15 points] Use the variational principle to estimate the deuteron ground-state energy.

Take a trial wavefunction \( \Psi = e^{-\alpha r/2a} \) where \( \alpha \) is a variational parameter. This is a 3D problem, but you can assume spherical symmetry; remember to use the proper form for the Laplacian in spherical coordinates. Note you will need to use the reduced mass of the neutron-proton system as the mass in your Hamiltonian. You may need to use a numerical or graphical method to minimize the variation.

b) [5 points] Compare your result to the experimental value for the deuteron binding energy, which is 2.22 MeV.

(over)
6. Consider a particle with mass $m$ in a two-dimensional square box (infinite potential walls) with sides of length $L$. There is a weak potential in the box given by

$$V(x, y) = V_0 L^2 \delta(x - x_0)(y - y_0)$$

a) [5 points] Determine the first-order correction to the ground state energy.

b) [10 points] Write the expression for the first-order estimate of the energy of the first excited state. For the special case that $x_0 = y_0 = L/4$, what is the splitting of this energy level?

c) [5 points] For what choices of the point $x_0, y_0$ would the first excited state remain degenerate, to this order in the perturbation? Explain this result in terms of the symmetry of the problem.