0. General Postulates of QM.

- A system has a state represented by an abstract vector $|\Psi\rangle$ psi $\Leftrightarrow$ pstate
- Between measurements, $|\Psi(t)\rangle$ evolves according to

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H \; |\Psi(t)\rangle$$

- Dynamical variables correspond to operators
- If we measure the dynamical variable having operator F, then
  - (i) the value we obtain is one of the eigenvalues of that operator F

    $$F\,|f_n\rangle = f_n\,|f_n\rangle$$
  - (ii) the proby of obtaining the value f_j is

    $$P_j = |\langle f_j|\Psi\rangle|^2$$
  - (iii) the process of measurement forces the system into the state $|\Psi\rangle = |f_j\rangle$, from whence it may continue to evolve.

[redacted]

I. The Vector Representing the State of a System: Forwards or Backwards in Time

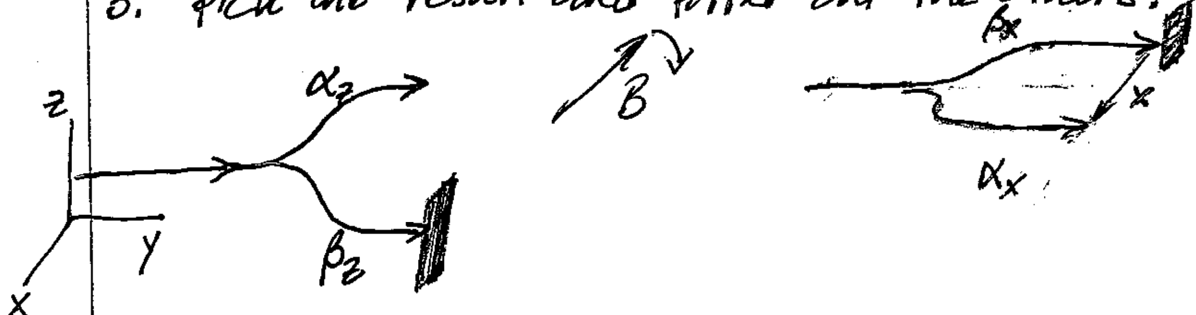
Examine a single spin-½ particle.

1. Choose a coordinate system $x\ y\ z$
2. Measure the component of spin S_z.

$$|\alpha_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \beta_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2}$$

3. Pick one result and filter out the others.



4. Act on the α's with various magnetic fields, pointing in various directions.

$$H = -\vec{\mu} \cdot \vec{B}(t) = +\frac{e}{m} \vec{S} \cdot \vec{B}$$

$$= \frac{e}{m} \left[S_x B_x(t) + S_y B_y(t) + S_z B_z(t) \right]$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

5. "The state of the system evolves under that Hamiltonian such that at any time t,

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H(t) |\Psi(t)\rangle \qquad |\Psi(t_i)\rangle = |\alpha\rangle$$

$$|\Psi(t)\rangle = U(t, t') |\Psi(t')\rangle \text{"}$$

" " mean that we think of that $|\Psi(t)\rangle$ as representing the actual, physical "wave function" or abstract vector of the system, the one that represents the state of the system at time t.

6. At some later time, measure a different component of spin, say S_x. Examine those measurements that come out S_x = "up" on the x axis, i.e. $S_x = +\hbar/2$. Throw away all other results.

$$\alpha_x = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \beta_x = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

The amplitude for finding this result is

$$\langle \alpha_x | \underbrace{U(t_f, t_i)\, \alpha_z}_{\text{instantaneous state of system}} \rangle$$

overlapped with final state.

7. It is also

$$= \langle U^{\dagger}(t_f, t_i)\, \alpha_x | \alpha_z \rangle = \langle U(t_i, t_f)\, \alpha_x | \alpha_z \rangle$$

8. Aharanov + Co.

Why do we ascribe "physical reality" to $U(t_f, t_i)|\alpha_z\rangle$ but not to $U(t_i, t_f)|\alpha_x\rangle$? Both are equally valid descriptions.

Since the final measurement produced $|\alpha_x\rangle$, we can say that at times leading up to that, the state of the system is represented by $U(t, t_f)|\alpha_x\rangle$

(This applies to those systems for which the initial state was found to be $|\alpha_z\rangle$ and the final state was found to be $|\alpha_x\rangle$.)

9. More generally if we have a well-defined initial state $|\psi\rangle$ and a known evolution operator, we say that the "state of the system is represented by
$$|\psi^+(t)\rangle = U(t, t_i)\,|\psi(t_i)\rangle"$$
and the amplitude for ending in my specified final state $|\phi(t_f)\rangle$ is
$$\langle\phi(t_f)|\cdot\psi^+(t_f)\rangle = \langle\phi(t_f)|U(t_f, t_i)\,\psi(t_i)\rangle$$
But this is equal to
$$\langle U^\dagger(t_f, t_i)\,\phi|\ \psi(t_i)\rangle$$
$$= \langle\phi^-(t_i)\ |\ \psi(t_i)\rangle$$
Why don't we say that the state just before the final measurement is represented by
$$\langle\phi^-(t)| = \langle U^\dagger(t_f, t)\,\phi(t_f)| = \langle U(t, t_f)\,\phi(t_f)|$$
$$= \text{bra conjugate to } U(t, t_f)\,|\phi(t_f)\rangle$$

We will be examining systems in which a "prior" measurement of a variable V is made with result v_n, <u>and</u> a "past" measurement of a variable W is made, with result w_m.

We will show that "weak measurements" made between those two events have unexpected results.

II. Von Neumann's Theory of Measurements by Entanglement

Suppose I want to measure a dynamical variable in a quantum system. The operator corresponding to that D.V. is $\hat{A}$, with eigenstates

$$A\,|\phi_a\rangle = a\,|\phi_a\rangle$$

The states $|\phi_a\rangle$ are a complete set in the Hilbert space of that dynamical system. The eigenvalues $\{a\}$ may be discrete or continuous.

Call that now system $\mathcal{A}$.

Example: $A = S_z$ in a spin ½ system. $a = \pm\frac{1}{2}\hbar$.

$$\phi_{1/2} = \begin{pmatrix}1\\0\end{pmatrix} \quad \phi_{-1/2} = \begin{pmatrix}0\\1\end{pmatrix} \quad A = \hbar/2 \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

To make the measurement, we must couple it to another system, the "measuring apparatus", $\mathbb{I}$.

Our measuring apparatus has a dynamical variable Q, having eigenvectors $|q\rangle$

$$Q\,|q\rangle = q\,|q\rangle$$

Let's suppose the spectrum is continuous (possibly bounded). We will call it a "position" operator, we will call the measuring apparatus a "pointer", and we presume that we can see the value of q, and we can control it (set its initial value).

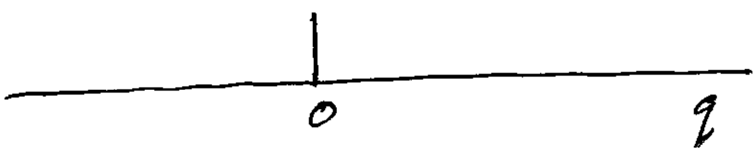
Q: How can we couple system + measuring apparatus such that by observing the pointer, we can infer the eigenvalue a?

The system [redacted] A is evolving according to its own Hamiltonian, H_A. [redacted]

First, let's presume that at some instant $\hat{t}$, system A is [redacted] definitely in one of the eigenstates, but we don't know which one.

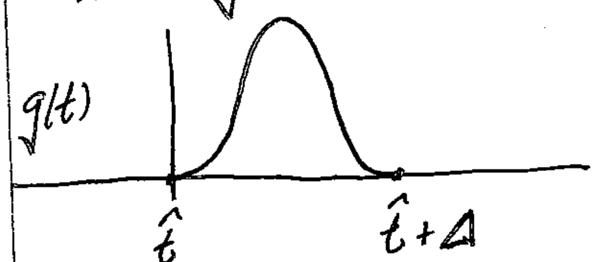
Let's arrange it so that at that instant, the "pointer" is in eigenstate $|0\rangle$.

$$\psi(q) = \delta(q-0).$$



To measure the value of a, apply for a short time a Hamiltonian to the combined system

$$H = g(t)\, P A$$



$$\int_{\hat{t}}^{\hat{t}+\Delta} g(t')\,dt' = 1$$

This H must strongly dominate during the time from $\hat{t}$ to $\hat{t}+\Delta$.

P is the momentum conjugate to position Q.

For that time, the evolution operator is

$$\exp\left[-i \int_{\hat{t}}^{t} g(t')\,dt'\; PA/\hbar\right]$$

and at $\hat{t}+\Delta$, $\quad U(\hat{t}+\Delta, \hat{t}) = \exp[-i PA/\hbar]$

Now

$$\exp[-iPA/\hbar] \qquad |a\rangle\,\delta(q-0)$$

$$= |a\rangle \exp[-i\,a\,(-i\hbar\tfrac{\partial}{\partial q})/\hbar] \quad \delta(q-0)$$

$$= |a\rangle \quad \exp(-\,a\tfrac{\partial}{\partial q}\,) \quad \delta(q-0)$$

$$= |a\rangle \quad \delta(q-a)$$

At the end, the pointer is in position $q = a$.

By observing the pointer, we have inferred the value of a.

Superpositions in $\mathcal{A}$.

Now suppose that the system $\mathcal{A}$ is in a superposition state:

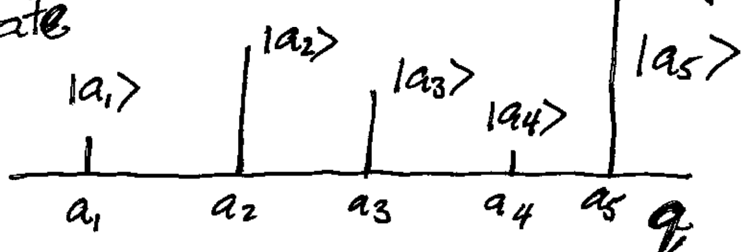
$$|\psi(t)\rangle = \sum_n c_n\, |a_n\rangle$$

Couple it to the "pointer", still assuming that the is initially in the state $\delta(q-0)$.

$$\exp[-iPA/\hbar]\left(\sum_n c_n |a_n\rangle\right) \delta(q-0)$$

$$= \sum_n c_n\, |a_n\rangle\, \delta(q-a_n) = \sum_n |a_n\rangle \quad c_n\, \delta(q-a_n)$$

The combined system of $\mathcal{A}$+pointer evolves to an entangled state



(note: cmplx coefficients)

Now we do the ideal measurement on the pointer.

q is forced into one of the eigenstates $\delta(q-a_n)$

we find a_n with probability $|c_n|^2$

and then system $\mathcal{A}$ must be in state $|a_n\rangle$

- Note that the <u>measurement</u> is a two-step process:
(i) entanglement with the pointer
(ii) observation of the pointer
Typically this last observation acts as a filter, so that afterwards the system is in ~~a~~ one particular state $|a\rangle$ and the pointer is in the state $\delta(q-a)$.

Statistical mixtures in $\mathcal{A}$.

Perhaps in setting up the system $\mathcal{A}$, we do not have an eigenstate or a pure superposition but a mixture. Then the "state" of $\mathcal{A}$ is described by a density op

$$\rho_{\mathcal{A}} = \sum_a p_a |a\rangle\langle a|$$

Suppose the state of the pointer is still $\delta(q-0)$. We will represent that as a density op for the pointer.

$$\rho_M = \delta(q-0)\,\delta(q'-0)$$

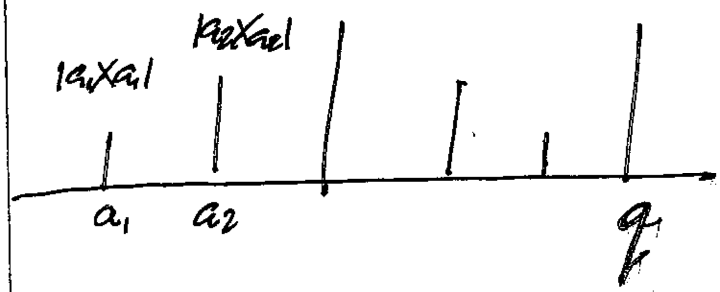
Apply the Hamiltonian, $g(t) P A$.

$$\rho_C(t+\Delta) = \exp(-iPA/\hbar)\ \rho_{\mathcal{A}}\,\rho_M\, \exp(+iPA/\hbar)$$

$$= \exp(-iPA/\hbar)\Big(\sum_a p_a |a\rangle\langle a|\Big)\ \delta(q-0)\,\delta(q'-0) \times \exp(+iPA/\hbar)$$

$$= \sum_a p_a |a\rangle \big[e^{-iaP/\hbar}\delta(q-0)\big]\ \delta(q'-0)\ e^{+iaP/\hbar}\langle a|$$

$$= \sum_a p_a |a\rangle\, \delta(q-a)\, \delta(q'-a)\, \langle a|$$



We find q at one of the a_n's, with proby p_n, and then the system A is in the pure state $|a_n\rangle\langle a_n|$.

Finally, suppose the pointer is in a superposition state near $q=0$.

e.g. $\delta(q-0) \longrightarrow N\ e^{-(q-0)^2/2\sigma^2} \equiv G(q-0)$

Then evolution takes it to the entangled state

$$\sum_a p_a\, |a\rangle\langle a| \quad G(q-a)\, G^*(q'-a)$$

Also if it is in a mixed state near $q=0$, the initial density op is

$$\left(\sum_a p_a\, |a\rangle\langle a|\right)\left(\sum_q p_q\, |q\rangle\langle q|\right)$$

It evolves to

$$\sum_a \sum_q p_a\, |a\rangle\langle a| \quad p_q\, |q-a\rangle\langle q-a|$$

and the sharp peaks are replaced by e.g. Gaussians.

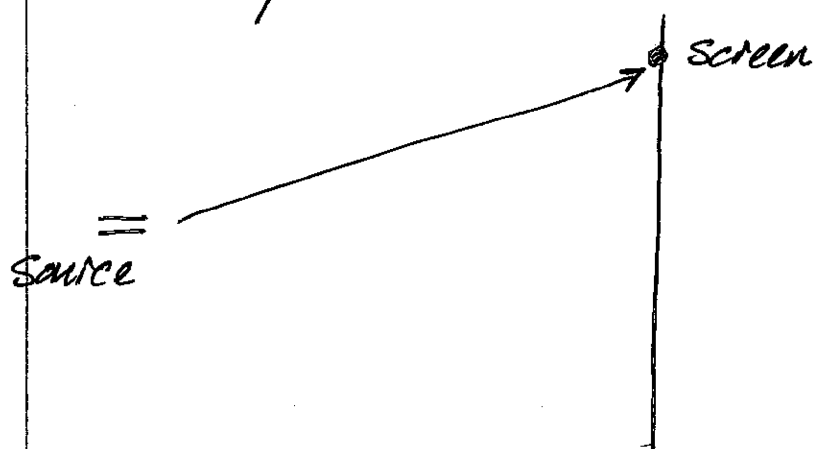
Note that the "pointer" never goes much past the largest eigenvalue.

Or does it?

We will soon study a case in which
- the largest eigenvalue is 100
- the width of the Gaussians is 10
- the pointer reads 140 (on average).

III. Stern Gerlach Case: Measurement by Entanglement

We cannot directly measure the spin of an atom. However we can measure the position of an atom, and from it we may be able to infer the momentum of the atom



Stern-Gerlach measurements entangle the spin vector of the atom with the momentum, thereby allowing the spin vector to be inferred from the momentum.

$$H = -\vec{\mu}\cdot\vec{B} = \varsigma\vec{S}\cdot\vec{B}$$

Suppose B is on the z axis, but it is inhomogeneous

$$\vec{B} = \hat{k}B_z(z) = \hat{k}\left(B_0 + z\frac{\partial B}{\partial z}\right)+\cdots$$

$$H = \Big[\underbrace{\varsigma S_z B_0}_{\text{precession only}} + \varsigma\frac{\partial B}{\partial z}\underbrace{z}_{\text{measurable}}\underbrace{S_z}_{\text{inferrable}}\Big]$$

$g(t)\,[\qquad\qquad]$

Let $P \equiv p_z$. $Q \equiv z$. The evolution operator is

$$\mathcal{U} = \exp[-i\ H\ (t_f - t_i)/\hbar]$$

Suppose the initial state is $\delta(P)\ |\alpha\rangle_z$.

That state evolves into

$$\exp\Big[-i/\hbar\ \underbrace{\int_{t_i}^{t_f} g(t')dt'\ 5\ \frac{\partial B}{\partial z}}_{2\Delta}\ Q S_z\Big]\ \delta(P)\ |\alpha_z\rangle$$

$$S_z\ |\alpha_z\rangle = +\frac{\hbar}{2}\ |\alpha_z\rangle$$

$$Q = +i\hbar\frac{\partial}{\partial P}$$

$$|\alpha_z\rangle \exp\left(\Delta \frac{\partial}{\partial P}\right)\ \delta(P) = |\alpha_z\rangle\, \delta(P + \Delta)$$

(positive S_z moves toward weaker fields)

The $|\beta_z\rangle$ state evolves into

$$|\beta_z\rangle\, \delta(P - P_0) \longrightarrow |\beta_z\rangle\ \delta(P - \Delta)$$

(negative S_z goes toward stronger fields).

IV. Surprising Quantum Effects

i	t_i	carry out a strong measurement of J_x on ~~...~~ a collection of N spins.
1	t_1	
2	t_2	
f	t_f	another strong measurement, this time of J_y

~~Suppose that~~ measurement i gives N $\qquad(\hbar = 2)$
and " f N

We want to know what will happen in <u>weak</u> measurements at t_1 + t_2.

my notation	their notation
P	q
Q	π
$H = g(t)\ PA$	$H = -g(t) q A$

Suppose the initial state of the pointer is uncertain as described earlier.

$\Delta P \sim N^{-1/2}$

$\Delta Q \sim N^{1/2}$ $\qquad N$ is large

$\Delta P \Delta Q = \hbar/2 \qquad = 1.$

We presume that at t_i, J_x has been measured and found to be N, and that at t_f, J_y has been measured and found to be N.

Now near t_1 make a "weak measurement" of J_x

$$A \rightarrow J_x$$

The system goes to

$$G(q-N) \quad |J_x = N\rangle \qquad \text{at } t_1 + \Delta$$

Near t_2 make a "weak measurement" of J_y

* We are only examining those systems in which the final J_y came out to be N. $\therefore$ we obtain the state

$$G(q-N) \quad |J_y = N\rangle$$

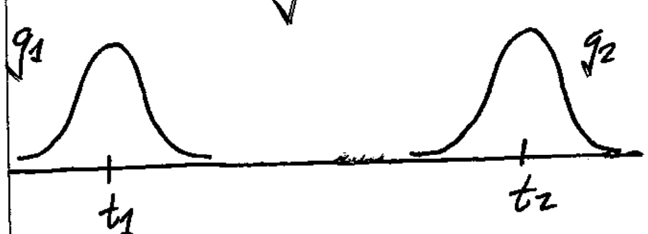
at time $t_2 + \Delta$.

~~This is equivalent to using~~

Now let's make ~~over~~ a modification of this as a "single measurement."

Define

$$H = g_1(t)\, P\, J_x/\sqrt{2} + g_2(t)\, P\, J_y/\sqrt{2}$$



The first makes the pointer move to $N/\sqrt{2}$; the second displaces it an equal distance. Afterwards, the needle is at $\sqrt{2}\,N$.

What happens if g_1 and g_2 overlap? ($t_1 = t_2$, but $\int(g_1+g_2)dt =$

Claim 2: the final position of the is the same, $\sqrt{2}\,N$.
Claim 1: this constitutes a measurement of $J_{\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)}$;

$$\mathcal{H} = g(t)\, P\left(\frac{J_x + J_y}{\sqrt{2}}\right). \qquad \int g(t')\,dt' = 1$$

Proof of claim 2:

The evolution op is

$$\exp - i\,\hat{P}\,(J_x + J_y)/\sqrt{2}$$

It acts upon the state

$$G(q-0)\ |J_x = N\rangle$$

We only examine those cases in which J_y turned out to be N.

$$F(q) = \langle J_y = N|\exp - iP\left(\frac{J_x+J_y}{\sqrt{2}}\right)|J_x = N\rangle\, G(q-0)$$

final state of needle

Proposition: We can evaluate this approximately because
width $G(q) \sim \sqrt{N}$
width $\hat{G}(p) \sim 1/\sqrt{N}$
and so in its action on G, the operator P is
$\sim$ bounded by $1/\sqrt{N}$.

$$\exp -i\frac{P}{\sqrt{2}}(J_x+J_y) = e^{-i\frac{P}{\sqrt{2}}J_y} \times e^{-\frac{P^2}{2}[J_x,J_y]\,\text{etc.}} \times e^{-i\frac{P}{\sqrt{2}}J_x}$$

$$e^{-(P^2/2)J_z}$$

$\sim$ bounded by $1/N$ — bounded by N

$$e^{\mathcal{O}(1)}$$

$$= e^{-i\frac{P}{\sqrt{2}}N}\; e^{-i\frac{P}{\sqrt{2}}N}\; e^{\mathcal{O}(1)}$$

$$= e^{-iP\sqrt{2}N}$$

$$F(q) = \langle J_{\cancel{x}} = N \mid J_x = N\rangle\; G(q-\sqrt{2}N)$$

The needle moves to a range of values larger than any eigenvalue!

eg: $N = 100$, $\sqrt{N} \sim 10$, $\sqrt{2}N \sim 140$

! The Expectation Value of the Position of the Pointer is far outside the range of eigenvalues!

PRL **60** 1351 (1988)
PRA **41** 11 (1990)

'How the Result of a Measurement of a Component of the Spin of a Spin-½ Particle Can Turn Out to be 100'

Previously:
- the observed property of the measuring apparatus was the position of the pointer Q
- Measure a quantity A by entangling it with Q

$$H_{int} = g(t)\, PA$$

$$\cancel{\frac{d\langle Q\rangle}{dt} = g(t)\,\langle [Q, PA]\rangle / i\hbar}$$

$$\cancel{= g(t)\,\langle A\rangle}$$

$$\cancel{\Delta\langle Q\rangle = \langle A\rangle}$$

If $|\Psi\rangle_{initial} = \sum_n c_n |a_n\rangle\, \delta(q-0)$
then

$$\Psi_{final} = \exp\left[-i\int g(t')\,dt'\, PA/\hbar\right]\Psi_{initial}$$

$$= \sum_n c_n\, |a_n\rangle\, \delta(q-a_n)$$

entangled state

Observation of q at one of the a_n's gives the value of that a_n, and typically the system is 'filtered' so that at that instant the state becomes $|a_n\rangle\,\delta(q-a_n)$.

Now (like Stern Gerlach)

- the observed quantity in measuring apparatus is ~~the~~ a momentum P
- the entangling Hamiltonian is

$$H = -g(t)\, Q A$$

Also, we understand that either

- we will repeat the expt many times after identical preparation, OR
- we will do it on a large number of identically-prepared systems

$$H = \sum_n -g(t)\, q_n A_n \qquad n \text{ labels "atoms"}$$

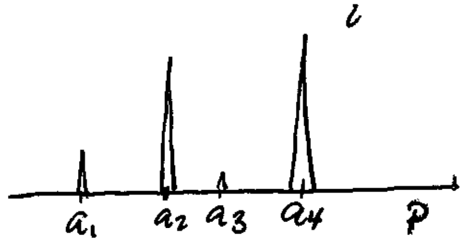
A perfect begins from a state

$$\delta(P-0) \sum_i c_i |a_i\rangle$$

[redacted] which evolves to

$$\sum_i c_i\, \delta(P-a_i)\, |a_i\rangle$$

(note: coefficients may be complex)



A WEAK MEASUREMENT begins from a state

$$\tilde{G}_{in}(P-0) = \frac{1}{(2\pi)^{1/4}(\Delta P)^{1/2}} \exp\left\{(P-0)^2/4(\Delta P)^2\right\} |\psi_{in}\rangle_n$$

the F.T. is

$$G_{in}(Q-0) = \frac{1}{(2\pi)^{1/4}}\frac{1}{(\Delta Q)^{1/2}} \exp\left[-(Q-0)^2/4(\Delta Q)^2\right] |\psi_{in}\rangle_n$$

$\Delta Q \Delta P = \hbar/2$ $\qquad$ Q is effectively bounded

Make a weak measurement of observable A between two strong measurements of some other observables. Then we know $|\Psi_{in}\rangle$ and $\langle\Psi_{final}|$.

Claim: the final wave fn of the pointer in the p-rep. can be

$$\tilde{G}(p) \simeq \exp\left[-(\Delta Q)^2 \left(p - \frac{\langle\Psi_f|A|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle}\right)^2\right] \quad N$$

a Gaussian centered on a strange value, called the weak value of A for the given initial and final states

$$A_w = \frac{\langle\Psi_f|A|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle}$$

Pf: $\langle\Psi_f| \exp\left[-i\int_t^{t+\Delta} g(t')dt' \quad QA/\hbar\right] |\Psi_i\rangle \quad G_{in}(Q)$

$= \langle f|\exp iQA|i\rangle \quad G_{in}(Q)$

$= \sum_n \langle f|A^n|i\rangle \frac{(iQ)^n}{n!} \quad G_{in}(Q)$

$= \langle f|i\rangle \sum_n (A^n)_w \frac{(iQ)^n}{n!} \quad G_{in}(Q)$

$= \langle f|i\rangle \sum_n \left\{(A_w)^n + \left[(A^n)_w - (A_w)^n\right]\right\} \frac{(iQ)^n}{n!} \quad G_{in}(Q)$

$$= \langle f|i\rangle \exp(iQA_w) \qquad G_{in}(Q)$$

$$+ \langle f|i\rangle \sum_{n=2}^{\infty} \frac{(iQ)^n}{n!}\left[(A^n)_w - (A_w)^n\right] \qquad G_{in}(Q)$$

If the Gaussian in Q is suf narrow (i.e. Gaussian in the measuring value P is suf wide), then the correction terms are small.

Since we have only been working on the operator, we may go back to the P rep

$$\langle \Psi_f|\, e^{-iH}\, |\Psi_i\rangle \qquad \tilde{G}_{in}(P)$$

$$= \langle f|i\rangle \exp(iQA_w) \quad \tilde{G}_{in}(P) \ + \text{corrections}$$

$$= \langle f|i\rangle \quad G_{in}(P - A_w)$$

A_w may be complex.
A Gaussian centered on a complex number is

$$\exp -(x-(a+ib))^2 = \exp[-(x-a)^2 + b^2]\, \exp[2ib(x-a)]$$

$$= \begin{pmatrix}\text{gaussian}\\ \text{around } a\end{pmatrix} \times [\cos + i\sin](2b(x-a))$$