Propagation of EM Waves through Anisotropic Materials

We will show that:

(i) $\underline{\underline{\varepsilon}}$ is a symmetric matrix, therefore has eigenvalues and orthogonal eigenvectors.
Those define "principal axes".
(ii) In these coordinates we write

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{pmatrix}$$

(iii) We consider the case $\varepsilon_x = \varepsilon_y$, so z axis is "special"

We will show that two types of waves propagate through the material

Ordinary wave: $\mathbb{E} \perp z$ index of refraction $n = \sqrt{\varepsilon_\perp}$

Extraordinary wave: $\mathbb{B} \perp z$

for this wave $\mathbb{E}$ is not $\perp$ to $\mathbb{k}$
index of refraction depends on direction of propagation.

Propagation of Electromagnetic Waves Through Anisotropic Materials

$\nabla \cdot \mathbb{D} = 4\pi \rho_{free}$ $\qquad\qquad$ $\nabla \cdot \mathbb{B} = 0$

$\nabla \times \mathbb{H} - \frac{1}{c}\partial \mathbb{D}/\partial t = \frac{4\pi}{c} J_{free}$ $\qquad$ $\nabla \times \mathbb{E} + \frac{1}{c}\partial \mathbb{B}/\partial t = 0$

$$\mathbb{D} = \underline{\underline{\varepsilon}}\ \mathbb{E}$$

Proposition: $\underline{\underline{\varepsilon}}$ is a symmetric matrix

$\longrightarrow E$

suppose $\mathbb{E}$ along x produces P at some angle, with some y component $P_y = \chi_{yx} E_x$

Then $\mathbb{E}$ along y must produce P with a corresponding x component $P_x = \chi_{xy} E_y$.

This is not an obvious property of materials.

Born and Wolf

CHAPTER XIV

OPTICS OF CRYSTALS

14.1. THE DIELECTRIC TENSOR OF AN ANISOTROPIC MEDIUM

IT will be remembered that our optical theory is based on two quite separate foundations; on the one hand on MAXWELL's equations § 1.1 (1) and (2), on the other hand on the material equations which in the case of an isotropic medium were given by the formulae § 1.1 (9)–(11). In dealing with crystals we must generalize these latter equations so as to take account of anisotropy. In the greater part of this chapter we assume that the medium is homogeneous, non-conducting ($\sigma = 0$), and magnetically isotropic,* but allow *electrical anisotropy*, i.e. we consider substances whose electrical excitations depend on the direction of the electric field. In general the vector $\boldsymbol{D}$ will then no longer be in the direction of the vector $\boldsymbol{E}$. In place of equation § 1.1 (10) we assume the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ to have the simplest form which can account for anisotropic behaviour, namely one in which each component of $\boldsymbol{D}$ is linearly related to the components of $\boldsymbol{E}$:

$$\left.\begin{aligned} D_x &= \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z, \\ D_y &= \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z, \\ D_z &= \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z. \end{aligned}\right\} \tag{1}$$

The nine quantities ε_{xx}, ε_{yy}, . . . are constants of the medium, and constitute the *dielectric tensor*; the vector $\boldsymbol{D}$ is thus the product of this tensor with $\boldsymbol{E}$.

We shall write equation (1) in shorter form as

$$D_k = \sum_l \varepsilon_{kl} E_l, \tag{2}$$

where k stands for one of the three indices x, y, and z, and l stands for each of x, y, and z in turn in the summation. The summation sign would be omitted in formal tensor notation, the occurrence of the index l in two places in the product being understood as an instruction to sum over all l's. We shall, however, retain the summation sign, as this will help to avoid any ambiguities for readers unfamiliar with tensor calculus.

We assume that the expressions § 1.1 (31) for electric and magnetic energy densities retain their validity. Thus

$$w_e = \frac{1}{8\pi}\,\boldsymbol{E}\,.\,\boldsymbol{D} = \frac{1}{8\pi}\sum_{kl} E_k \varepsilon_{kl} E_l. \tag{3}$$

---

* There are also magnetic crystals, but as the effect of magnetization on optical phenomena (rapid oscillations) is small, the magnetic anisotropy may be neglected. The magnetic permeability shall, however, be retained, and will be represented by a scalar μ, in order to preserve some symmetry in the formulae and to include weakly magnetic crystals; moreover, by retaining it, we facilitate the expression of the equations in systems of units in which μ is not equal to unity in free space.

and

$$w_m = \frac{1}{8\pi} \boldsymbol{B} \cdot \boldsymbol{H} = \frac{1}{8\pi} \mu H^2. \tag{4}$$

We retain also the definition § 1.1 (38) of the POYNTING vector, or the "ray-vector"

$$\boldsymbol{S} = \frac{c}{4\pi} (\boldsymbol{E} \wedge \boldsymbol{H}) \tag{5}$$

and investigate whether these definitions are consistent with the principle of the conservation of energy.

We have, as in § 1.1.4, by multiplying the first MAXWELL equation by $\boldsymbol{E}$ and the second by $\boldsymbol{H}$ and using the vector identity § 1.1 (27),

$$\begin{aligned} -c \operatorname{div} (\boldsymbol{E} \wedge \boldsymbol{H}) &= \boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}} \\ &= \sum_{kl} E_k \varepsilon_{kl} \dot{E}_l + \frac{1}{2} \frac{d}{dt} (\mu H^2). \end{aligned} \tag{6}$$

If we divide both sides of this equation by 4π, the second term on the right represents the rate of change of the magnetic energy per unit volume, but the first term does not represent the rate of change of the electric energy density unless

$$\frac{1}{4\pi} \sum_{kl} E_k \varepsilon_{kl} \dot{E}_l = \frac{dw_e}{dt} = \frac{1}{8\pi} \sum_{kl} \varepsilon_{kl} (E_k \dot{E}_l + E_l \dot{E}_k) \tag{7}$$

that is, unless

$$\sum_{kl} \varepsilon_{kl} (E_k \dot{E}_l - \dot{E}_k E_l) = 0.$$

The suffices k and l are dummy suffices; both run over the same values (x, y, z). Hence the expression is not altered if we interchange k and l in the second term. This leads to

$$\sum_{kl} (\varepsilon_{kl} - \varepsilon_{lk}) E_k \dot{E}_l = 0.$$

As this equation must hold whatever the value of the field, it follows that

$$\varepsilon_{kl} = \varepsilon_{lk}. \tag{8}$$

This means that the *dielectric tensor must be symmetric*; it has only six instead of nine independent components. Conversely, the condition (8) is sufficient to ensure the validity of equation (7), and we obtain the energy theorem in differential form (the "hydrodynamical continuity equation" § 1.1 (43))

$$-\operatorname{div} \boldsymbol{S} = \frac{dw}{dt}, \quad (w = w_e + w_m). \tag{9}$$

The symmetry of the tensor ε makes it possible to reduce the expression for the electric energy w_e to a form in which only the squares of the field components, and not their products, enter. Consider in a space x, y, z the surface of the second degree

$$\varepsilon_{xx} x^2 + \varepsilon_{yy} y^2 + \varepsilon_{zz} z^2 + 2\varepsilon_{yz} yz + 2\varepsilon_{xz} xz + 2\varepsilon_{xy} xy = \text{const.} \tag{10}$$

The left-hand side of (10) must be a positive definite quadratic form, because if x, y, and z are replaced by the components of $\boldsymbol{E}$ the expression becomes equal to $8\pi w_e$, and the energy w_e must be positive for any value of the field vector. Therefore equation (10) represents an ellipsoid. The ellipsoid can always be transformed to its principal axes; thus there exists a coordinate system fixed in the crystal such that the equation of the ellipsoid is

$$\varepsilon_x x^2 + \varepsilon_y y^2 + \varepsilon_z z^2 = \text{constant}. \tag{11}$$

In this system of *principal dielectric axes* the material equations and the expression for the electrical energy take the simple forms

$$D_x = \varepsilon_x E_x, \qquad D_y = \varepsilon_y E_y, \qquad D_z = \varepsilon_z E_z, \tag{12}$$

$$\begin{aligned} w_e &= \frac{1}{8\pi}(\varepsilon_x E_x^2 + \varepsilon_y E_y^2 + \varepsilon_z E_z^2) \\ &= \frac{1}{8\pi}\left(\frac{D_x^2}{\varepsilon_x} + \frac{D_y^2}{\varepsilon_y} + \frac{D_z^2}{\varepsilon_z}\right). \end{aligned} \tag{13}$$

ε_x, ε_y, ε_z are called the *principal dielectric constants* (or *principal permittivities*). It may be seen immediately from these formulae that $\boldsymbol{D}$ and $\boldsymbol{E}$ will have different directions, unless $\boldsymbol{E}$ coincides in direction with one of the principal axes, or the principal dielectric constants are all equal; in the latter case ($\varepsilon_x = \varepsilon_y = \varepsilon_z$) the ellipsoid degenerates into a sphere.

A note must be added here on the effect of dispersion. Just as, in the case of isotropic substances, the dielectric constant is not a constant of the material but depends on the frequency, so in an anisotropic medium the six components ε_{kl} of the dielectric tensor will also vary with frequency. As a result not only the values of the principal dielectric constants ε_x, ε_y, ε_z will vary but also the directions of the principal axes. This phenomenon is known as *dispersion of the axes*. It can arise, however, only in crystals in which the symmetry of the structure does not determine a preferential orthogonal triplet of directions, i.e. it can be observed only in monoclinic and triclinic systems (cf. § 14.3.1).*

If we restrict ourselves to monochromatic waves we may disregard dispersion; the quantities ε_{kl} are then constants depending only on the medium.

14.2. THE STRUCTURE OF A MONOCHROMATIC PLANE WAVE IN AN ANISOTROPIC MEDIUM

14.2.1 The phase velocity and the ray velocity

In a monochromatic plane wave of angular frequency $\omega = 2\pi\nu$ propagated with velocity c/n in the direction of the unit wave-normal $\boldsymbol{s}$, the vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, and $\boldsymbol{B}$ are in complex notation proportional to $\exp\left[i\omega\left(\frac{n}{c}(\boldsymbol{r}.\boldsymbol{s}) - t\right)\right]$. It may be mentioned at once that in addition to the *phase* (or *wave-normal*) *velocity* c/n, we shall later have to introduce a *ray* (or *energy*) *velocity*, since, as will be seen, in an anisotropic medium the energy is in general propagated with a different velocity and in a direction different from that of the wave normal.

In such an oscillatory field the operation $\partial/\partial t$ is always equivalent to multiplication

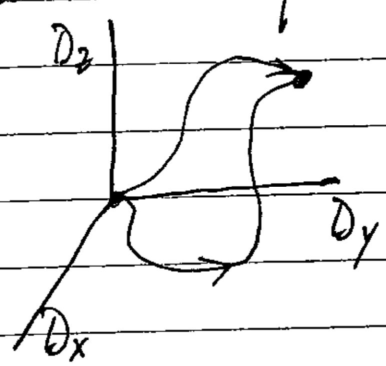
* This dispersion phenomenon is particularly conspicuous in the infra-red: cf. TH. LIEBISCH and H. RUBENS, *Sitzber. preuss. Akad. Wiss., Phys.-math. Kl.* (1919), 198, 876. H. RUBENS, *ibid.*, 976

Proof:

1. If we change applied field on a dielectric then the local energy density $U(\vec{r}) = \frac{\text{energy}}{\text{unit volume}}$ of charges and fields in the dielectric changes by

$$dU = \mathbb{E} \cdot d\mathbb{D}$$

2. This must be an exact differential. The energy density must depend only on the fields, not on the path by which we applied these fields.

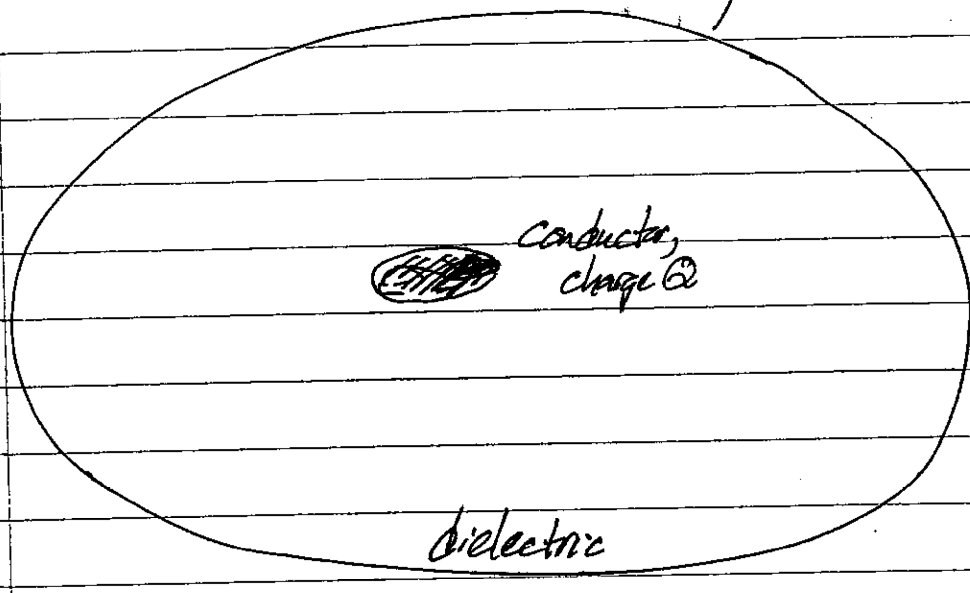


3. The condition for an exact differential is

$$\frac{\partial E_x}{\partial D_y} = \frac{\partial E_y}{\partial D_x} \quad \text{etc.}$$

But since $\mathbb{D} = \underline{\varepsilon}\,\mathbb{E}$, $\mathbb{E} = \underline{\varepsilon}^{-1}\,\mathbb{D}$, so ε^{-1} must be sym. $\therefore$ ε is sym.

Proof of 1. From L+L Electrodynamics of Continuous Media



$\nabla \cdot \mathbf{D} = \rho$

$\int \mathbf{D} \cdot \mathbf{n} \, dS = Q$ $\qquad$ n = outward pointing normal

Potential of conductor $= \phi$

Now add a little charge to the conductor, δQ.
Work required to do this is

$$\delta W = \phi_{cond} \, \delta Q$$
$$= \int \phi_{conductor} \, \mathbf{D} \cdot n \, dS$$
$$= \int \phi(\mathbf{r}) \, \mathbf{D} \cdot n \, dS$$
$$= -\int \phi(\mathbf{r}) \, \mathbf{D} \cdot n' \, dS$$

Now change to a different volume integral: over all space outside the conductor.
$\mathbb{D} \to 0$ as $1/R^2$
$\phi \to 0$ as $1/R$

$$\therefore \oint_{sf\ \text{at}\ \infty} = 0$$

$$\delta W = -\int_{\text{all space outside conductor}} \nabla\cdot\left[\phi\, \delta \mathbb{D}\right]\, dV$$

$$= +\int -(\nabla \phi)\cdot \delta \mathbb{D} + \phi\underbrace{(\nabla\cdot \delta \mathbb{D})}_{=0}\, dV$$

$$= \int \mathbb{E}\cdot \delta \mathbb{D}\, dV$$

The work done $= \int \left[\text{Change in energy density}\right] dV$

$$dU = \mathbb{E}\cdot d\mathbb{D}$$

Note: this argument is reversible. Work input to add charge = work output if charge is taken away. $\therefore$ it assumes none of the work is converted to heat. This is valid if charge is added slowly, so $\mathbb{P}(\mathbf{r},t) = \int \chi(t-t')\, \mathbb{E}(\mathbf{r},t')\, dt'$

$$= \chi\, \mathbb{E}(\mathbf{r},t)$$

There fore there exists a real orthogonal transformation which makes $\underline{\underline{\varepsilon}}$ diagonal.

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{x'} & 0 & 0 \\ 0 & \varepsilon_{y'} & 0 \\ 0 & 0 & \varepsilon_{z'} \end{pmatrix}$$

These are called the principal axes of the material.

Mills examines the case in which $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$

How does a plane wave propagate thru this material?

The wave must be
$$\begin{aligned} \mathbb{D}(\mathbb{r},t) &= \mathbb{D}_0\, e^{i(\mathbb{k}\cdot\mathbb{r}-\omega t)} \\ \mathbb{E} \quad " &= \mathbb{E}_0 \quad " \\ \mathbb{B} \quad " &= \mathbb{B}_0 \quad " \\ \mathbb{H} \quad " &= \mathbb{H}_0 \end{aligned}$$

Then since $\rho_{free} = 0 = \mathbb{J}_{free}$

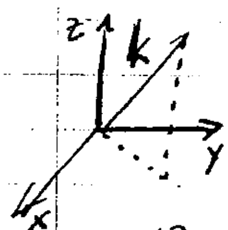
$$\nabla \cdot \mathbb{D} = \mathbb{k} \cdot \mathbb{D}_0 = 0$$
$$\nabla \cdot \mathbb{B} = \mathbb{k} \cdot \mathbb{B}_0 = 0$$

$$\nabla \times \mathbb{E} + \frac{1}{c}\frac{\partial \mathbb{B}}{\partial t} = \mathbb{k} \times \mathbb{E}_0 - \frac{\omega}{c}\mathbb{B}_0 = 0$$

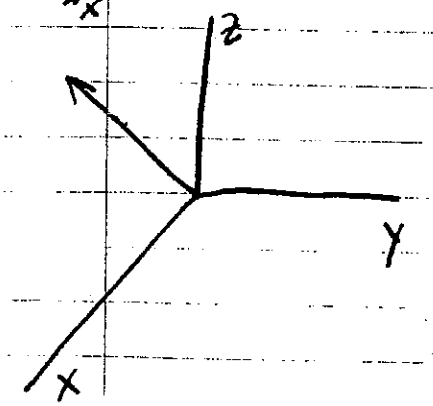
$$\nabla \times \mathbb{H} - \frac{1}{c}\frac{\partial \mathbb{D}}{\partial t} = \mathbb{k} \times \mathbb{H}_0 + \frac{\omega}{c}\mathbb{D}_0 = 0$$

$\Rightarrow$ $\mathbb{D}_0$ and $\mathbb{B}_0$ are perpendicular to $\mathbb{k}$.

$\Rightarrow$ $\mathbb{E}$ is NOT necessarily perpendicular to $\mathbb{k}$.



For general $\mathbb{k}$, we can rotate in the xy plane so that the new x axis aligns with $\mathbb{k}$: i.e. $k_y = 0$.



$\mathbb{D}_0 = \underline{\underline{\varepsilon}}\ \mathbb{E}_0 \qquad \mathbb{H}_0 = \mathbb{B}_0$ in our units

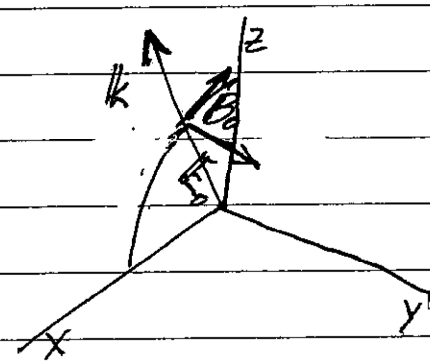
Ordinary Wave

$$\mathbb{E}_0 = \hat{j}\, E_y$$

$$\mathbb{k} \times \mathbb{E}_0 = \frac{\omega}{c}\, \mathbb{B}_0$$

i.e.

$$\mathbb{B}_0 = \frac{c}{\omega}\left(\mathbb{k} \times \mathbb{E}_0\right)$$



$\mathbb{B}_0$ is perp to both $\mathbb{E}_0$ and $\mathbb{k}$

$$\mathbb{D}_0 = -\frac{c}{\omega}\, \mathbb{k} \times \mathbb{B}_0$$

$\mathbb{D}_0$ is perp to both $\mathbb{k}$ and $\mathbb{B}_0$. It must be parallel to $\mathbb{E}_0$

Is that consistent?

$$\mathbb{D}_0 = \underline{\underline{\varepsilon}}\ \mathbb{E}$$

i.e. $D_y = \varepsilon_y E_y$ direction is consistent.

magnitude is consistent if

$$D_y = \varepsilon_y E_y = \frac{c^2}{\omega^2} k^2 E_y$$

i.e. if

$$\omega = \frac{c}{\sqrt{\varepsilon_y}}\, k$$

An ordinary wave propagating in a medium with $n = \sqrt{\varepsilon_y}$.

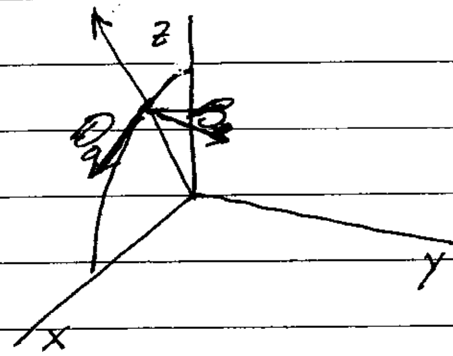
(Note also $\mathbb{k} \cdot \mathbb{D}_0 = \mathbb{k} \cdot \mathbb{B}_0 = 0$ automatically.

An 8'

Extraordinary wave

$$\mathbb{B}_0 = \hat{j}\, \mathbb{B}_y$$

$$\mathbb{D}_0 = -\frac{c}{\omega}\, \mathbb{k} \times \mathbb{B}_0$$



$$\mathbb{E}_0 = \varepsilon^{-1}\, \mathbb{D}_0$$

$$\begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon_\perp} D_x \\ 0 \\ \frac{1}{\varepsilon_\parallel} D_z \end{pmatrix}$$

$\mathbb{E}$ lies in xz plane but is <u>not</u> parallel to $\mathbb{D}$.

$$\frac{c}{\omega}\, \mathbb{k} \times \mathbb{E} = \mathbb{B}_0$$

Consistent if $\frac{c}{\omega}(k_z E_x - k_x E_z) = B_y$

$$\frac{c}{\omega}\left(\frac{k_z}{\varepsilon_\perp} D_x - \frac{k_x}{\varepsilon_\parallel} D_z\right) =$$

$$\frac{c^2}{\omega^2}\left(\frac{k_z^2}{\varepsilon_\perp} + \frac{k_x^2}{\varepsilon_\parallel}\right) B_y = B_y$$

$$D_x = -\frac{c}{\omega}(k_y B_z - k_z B_y)$$

$$D_z = -\frac{c}{\omega}(k_x B_y - k_y B_x)$$

i.e. consistent if

$$\omega^2 = c^2\left(\frac{k_z^2}{\varepsilon_\perp} + \frac{k_x^2}{\varepsilon_\parallel}\right)$$

Clearly the effective index of refraction depends on direction:

Write $k_z^2 = k^2 \cos^2\theta \quad k_x^2 = k^2 \sin^2\theta$

$$\frac{\omega^2}{k^2} = c^2 \left(\frac{\cos^2\theta}{\varepsilon_\perp} + \frac{\sin^2\theta}{\varepsilon_\parallel} \right)$$

$$n = n(\theta) = \left[\frac{1}{\frac{\cos^2\theta}{\varepsilon_\perp} + \frac{\sin^2\theta}{\varepsilon_\parallel}} \right]^{1/2}$$

Later we will show that to get period doubling we must match phase velocities

$$v_\phi(\omega) = v_\phi(2\omega)$$

In general this does not happen.

However, suppose $\varepsilon_{\parallel}(2\omega) < \varepsilon_{\perp}(\omega)$

Then