Running of the Coupling in AdS/QCD

A thesis submitted in partial fulfillment of the requirement for
the degree of Bachelor of Science in Physics from
The College of William and Mary

by

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Accepted for: B.S. in Physics

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Williamsburg, VA
May 2009

Abstract

We present here a method of improving the Anti de Sitter/Quantum Chromodynamics (AdS/QCD) model of Quantum Chromodynamics (QCD). AdS/QCD is a model of the strong force which is approximately conformal for energies larger than the mass of the rho meson; that is, the coupling of the strong force is constant at high energy scales. QCD is in fact non-conformal, although it approaches conformality in the high energy regime; however, the low energy predictions of AdS/QCD are accurate to approximately 10% when compared with experimental data. We attempt to build non-conformality into AdS/QCD by crudely accounting for the running of the coupling. This is done by altering the AdS geometry used in the model.

1 Introduction and Background

AdS/QCD is a model that is used to approximate the behavior of the particles and interactions of quantum chromodynamics. This model is based on the principles of the AdS/CFT correspondence. A Conformal Field Theory (CFT) is a field theory which behaves similarly at all length or energy scales, that is, the coupling between particles is independent of the distance between the particles. The AdS/CFT correspondence is a relationship between a $(3+1)$ dimensional CFT without gravity and a $(4+1)$ dimensional field theory in an Anti de Sitter (AdS) spacetime background, with gravity [3]. The correspondence between these theories depends on the fact that the conformal symmetry in a $(3+1)$ dimensional CFT has the same group structure as the set of isometries (transformations which leave the metric invariant) on $(4+1)$ dimensional AdS. Because the interactions in the theory in AdS are weakly coupled, complex problems in CFT may be solved more easily using pertubative methods in this model.

AdS/QCD is a model based on the AdS/CFT correspondence which approximates the behavior of particles of QCD using $(4+1)$ dimensional AdS. It is seen that this theory provides good approximations to the observed behaviors of the particles of QCD. This theory is conformal at high energy scales, and non-conformal at low energy scales. QCD, however, is non-conformal even at high energy scales. It is the goal of this project to build into this model non-conformality (or running of the coupling), and to determine how well this new model predicts the properties of particles in QCD. In order to model this running of the coupling in AdS/QCD, we alter the geometry of AdS slightly, allowing it to vary as a function of the extra dimension.

2 Theory

Classical theories of particle interactions assume that the coupling between interacting particles remains constant at all length scales. For example, in classical electrodynamics, the electrostatic force between two charged particles is proportional to the product of the charges and inversely proportional to the square of the distance between them. This force is then described using a single proportionality factor, which is assumed to remain constant as the distance between the particles varies. However, it is seen that at extremely small length scales, this coupling factor is not constant; it increases as the distance between the particles decreases. The opposite effect is seen in quantum chromodynamics. The strong coupling increases with distance, and for large enough distances, the force acting between particles does not change with distance. This is what gives rise to quark-gluon confinement in QCD. As two interacting quarks are pulled apart, the strong field between them does not spread out, but rather forms a “flux tube” between the quarks. The energy contained in this flux tube increases linearly with distance. If two interacting quarks are spread far enough apart, the energy contained in the flux tube will generate a quark and an antiquark, breaking the flux tube and resulting in two separate quark-antiquark pairs.

AdS/QCD attempts to reformulate QCD as a string theory in $(4+1)$ dimensional AdS space. Interactions between a quark and an antiquark are modeled as an open string terminating at the quarks, and extending into the extra dimension of Anti de Sitter space. We consider a one-dimensional “slice” of the space in the extra dimension, labeled by the variable z, and apply two boundaries to it. The ultraviolet (UV), or high energy, boundary is placed at $z = \epsilon$, where epsilon is taken to approach zero. The reason this is done is because the metric of AdS contains a factor of $1/z$, and infinitesimal distances approach infinity as z approaches zero. The infrared (IR), or low energy, boundary is placed at some value of z, which we call $z = z_m$. This upper boundary assures that the theory will be confining. If we consider two interacting quarks placed at the UV boundary, the string connecting them will extend into the space (Figure 1). The string will follow a path in the space defined by the geodesic of AdS between the two quarks, defined by the metric of AdS:

$$ds^2 = \frac{R^2}{z^2}(dx^\mu dx^\nu \eta_{\mu\nu} - dz^2), \tag{1}$$

where $\eta_{\mu\nu}$ is a 4×4 diagonal tensor given by $\text{diag}(1, -1, -1, -1)$. As the quarks are separated, the string will increase in length and extend farther and farther into the z-dimension of the space. The energy contained in the fields in QCD is related to the length of the string in AdS. For short enough strings, as the string increases in length, the force of attraction between the particles decreases with the square of their distance; the attraction is thus a Coulomb attraction, and the coupling factor remains unchanged. Once the string extends far enough into the z-dimension, it reaches the IR boundary and cannot extend any further in that dimension. It extends linearly along the boundary (orthogonal to the z direction), while the segments of string connecting the boundary to the quarks remain unchanged in shape (Figure 2). Due to this, the length of the string increases linearly, as do the energy contained in the fields and the coupling factor. Because the coupling factor does not change when the quarks are relatively close together, AdS/QCD is conformal in this regime.

The first part of this project involves verifying some known results of AdS/QCD. Bound states of particles in QCD, such as ρ-mesons, can be modeled as Kaluza-Klein modes of gauge fields in the z-dimension of AdS. This is similar to finding the wave function of a particle in an infinite square potential well of one dimension; certain boundaries are specified, and the function must meet certain conditions at the boundaries. In the case of the particle in a box, the boundary conditions are Dirichlet boundary conditions; the function must go to zero at the boundaries. In AdS/QCD, the boundaries are the UV and IR boundaries mentioned before, and one boundary condition is a Dirichlet, and the other a Neumann boundary condition;

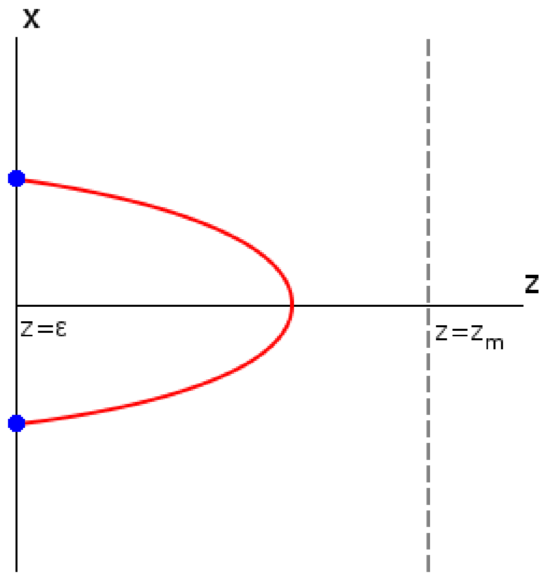


Figure 1: A string extending into the z dimension of a slice of AdS. The blue circles represent a quark-antiquark pair, while the red curve represents the string connecting them. Here, x labels the other three spatial dimensions.

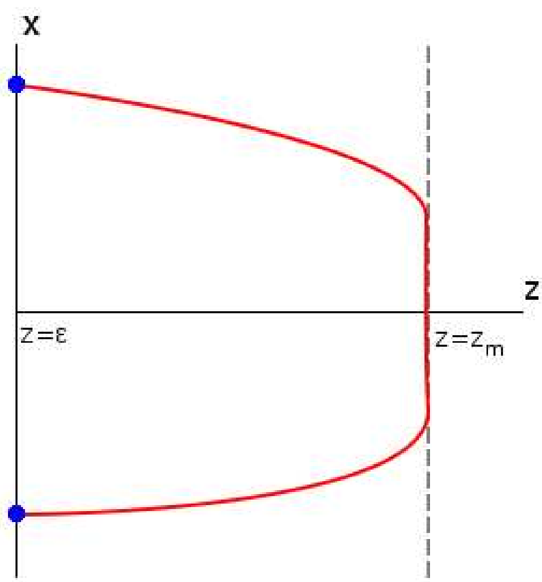


Figure 2: The string extends linearly along the IR boundary at $z = z_m$.

the gauge field must go to zero at the UV boundary, and its first derivative with respect to z must go to zero at the IR boundary. These specific conditions are chosen since the IR boundary condition must be gauge invariant, and these are the simplest gauge invariant boundary conditions. We want to verify the masses of ρ mesons as predicted by AdS/QCD. The differential equation describing the transverse (z direction) component of the gauge field for ρ mesons is [2] [4]

$$\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}V_\mu^a(q,z)\right)+\frac{q^2}{z}V_\mu^a(q,z)=0, \tag{2}$$

where $V_\mu^a(q,z)$ is the four dimensional Fourier transform of $V_\mu^a(x,z)$, which is the vector field, and q is proportional to the mass of the ρ meson. This is an eigenvalue problem, which has as its solutions a countably infinite number of q values corresponding to an infinite tower of ρ mesons. In reality, ρ mesons are detected as peaks in a scattering cross section with energies corresponding to the ρ meson masses. It is observed that at higher energies, the peaks get wider, and at some point, they are so wide as to no longer be distinguishable as mesons; therefore, there does not actually exist an infinite tower of ρ mesons.

We may also, for example, verify the value of the ρ meson decay constant, which determines the electromagnetic rate of decay of the ρ meson and thus its stability. The ρ meson decay constant is given by [2]

$$F_\rho^2=\frac{1}{g_5^2}[\psi_\rho'(\epsilon)/\epsilon]^2=\frac{1}{g_5^2}[\psi_\rho''(0)]^2, \tag{3}$$

where ψ_ρ is a solution to Equation (2), and g_5^2 is given by [2]

$$g_5^2=\frac{12\pi^2}{N_c}. \tag{4}$$

Here, N_c is the number of colors in QCD (three); thus $g_5=2\pi$.

The second part of this project involves altering the model in order to account for the running of the coupling observed in QCD. This is done by altering the geometry of the slice of AdS used in the model, allowing its curvature to vary as a function of the extra dimension z. It is observed that QCD is not conformal; by varying the geometry we may build this non-conformality into the model.

3 Methods

Equation (2) is solved numerically using the numerical differential equation solver in *Mathematica*. This is done via separation of variables: we assume that our solutions are of the form

$$V_\mu^a(q,z) = V_{\mu n}^a(q)\psi_n(z), \tag{5}$$

where the subscripts n label the eigenfunctions which solve equation (2). With this ansatz, Equation (2) yields an equation for $\psi_n(z)$:

$$\frac{1}{z}\left(\frac{\mathrm{d}^2\psi_n}{\mathrm{d}z^2} - \frac{1}{z}\frac{\mathrm{d}\psi_n}{\mathrm{d}z} + q^2\psi_n\right) = 0, \tag{6}$$

subject to the boundary conditions

$$\psi_n(\epsilon) = 0, \qquad \left.\frac{\mathrm{d}\psi_n}{\mathrm{d}z}\right|_{z=z_m} = 0. \tag{7}$$

Once a given eigenvalue q is found, we may solve for the ρ decay constant in Equation (3).

This procedure amounts to a Kaluza-Klein decomposition of the 5D gauge fields. Once these results were obtained and use to verify the predictions of AdS/QCD, the model was altered and its new predictions tested. A more accurate method of including running of the coupling would allow the geometry of AdS to vary continuously as a function of z; here, we present a simpler method of achieving this goal. We define the geometry of the slice of AdS in two regions: From $z = \epsilon$ up to some point to be determined, called z_0, we allow the space to retain its original AdS curvature. At $z = z_0$, the curvature changes abruptly and remains constant from $z = z_0$ to $z = z_m$. Mathematically, this is represented by changing the factors of $1/z$ in both terms of Equation (2). For $z < z_0$, it remains $1/z$, and for $z > z_0$, we use

$$\frac{1}{z} \to \frac{r}{z - (1 - r)z_0}, \tag{8}$$

where r is some constant which determines the ratio of the curvatures in the two regions. The term $-(1-r)z_0$ in the denominator here ensures that the metric remains continuous at $z = z_0$.

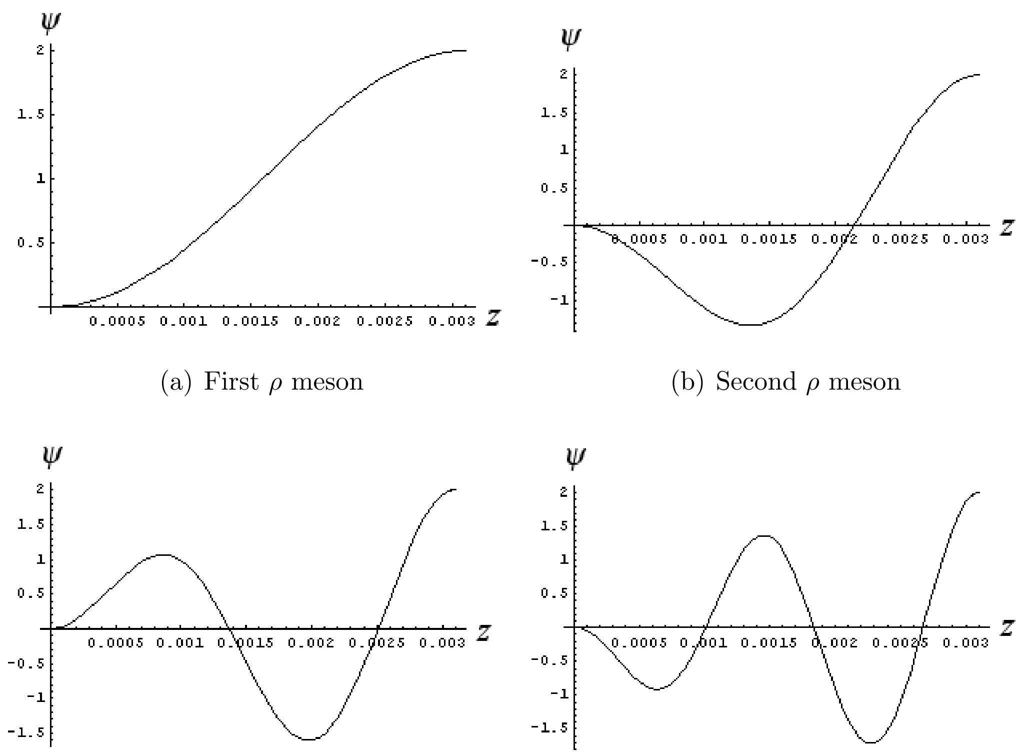
Our goal now is to attempt to determine r and z_0 in order to improve the predictions of this model.

4 Results

Using *Mathematica*, the first few eigenvalues of Equation (6) were found. These values were $q_1 = 2.4048$, $q_2 = 5.5201$, $q_3 = 8.6537$, and $q_4 = 11.792$. It should be noted that these are approximately the first four zeros of the J_0 Bessel function. The solutions found by this method are not normalized, since the scale of the solutions depends on our choices of ϵ, z_m, and $\mathrm{d}f/\mathrm{d}z$ at ϵ. In order to determine the proper value of z_m, we use the relation

$$z_m = \frac{q}{m_\rho}, \tag{9}$$

where m_ρ is the mass of the ρ meson corresponding to the eigenvalue q. The mass of the first ρ meson is known to be 776 MeV [1] (all values stated in this section are given in natural units; $c = \hbar = 1$). Using this value, along with $q = 2.4048$, in the above equation, we obtain $z_m = 3.10 \times 10^{-9}$ eV^{-1}. This produces masses of the second, third, and fourth ρ mesons of 1.78 GeV, 2.79 GeV, and 3.81 GeV, respectively. This method of determining ρ meson masses is only valid for low energies; the experimental values of the second, third, and fourth ρ masses are 1.45 GeV, 1.70 GeV, and 2.1 GeV, respectively. As the energy of the mesons increase, further corrections are needed. The z-dimension gauge fields are shown in Figure 3 below.



(a) First ρ meson

(b) Second ρ meson

(c) Third ρ meson

(d) Fourth ρ meson

Figure 3: Wavefunctions of the first four ρ mesons

Using Equation (3) with the wave function of the lightest ρ meson, we find that $F_\rho^{1/2} = 329$ MeV. This agrees with the value for the unaltered model stated in [2].

In order to make useful predictions about whether or not this model can be improved via the methods described above, we must be able to calculate a number of quantities which can be predicted by this new model. Since this model involves two free parameters, r and z_0, we must calculate at least two observables. Knowing one observable, we can determine r in terms of z_0 under the condition that this observable take on its experimental value. We can then fix specific values of r and z_0 such that the second observable is calculated to be as close to its experimental value as possible. Once these values of r and z_0 are fixed, we may use them to calculate the values of many other observables using this model, which will allow us to determine whether or not this new model is a significant improvement over the old one [2]. We have managed to calculate two observables using this model, namely the ρ meson mass and the ρ meson decay constant. Table 1 shows some values of r and z_0 which produce a value of $F_\rho^{1/2}$ of approximately 345 MeV (its observed value [2]), along with the values of the second ρ meson mass for these values of r and z_0. As stated before, the experimental value of the mass of the second ρ meson is 1,450 MeV. Since the original model produces a second ρ meson mass of 1,780 MeV, it appears as though this new model produces results less accurate than the original. In this model, it was observed that for values of r less than one, no values of z_0 would produce a value of $F_\rho^{1/2}$ of approximately 345 MeV. For values of z_0 less than z_m, the value of $F_\rho^{1/2}$ is significantly less than 345 MeV; for values of z_0 greater than z_m, $F_\rho^{1/2}$ simply takes on its value from the original, unaltered model. This is due to the fact that changes to the AdS beyond the IR boundary do not affect our model; the string connecting a quark-antiquark pair does not extend beyond the IR boundary. Since we have defined that the AdS metric would remain unchanged from the previous model for $z < z_0$, the metric for the entire slice of AdS remains what it was in the previous model, when $z_0 > z_m$.

There are some other interesting quantities which may be calculated using this model. For example, one may determine the IR boundary, z_m, as a function of the geometry ratio, r, for a given geometry boundary, z_0. This function is shown in figure 4 for the geometry boundary $z_0 = 1/400\ \text{MeV}^{-1}$.

r	$1/z_0$ (MeV)	$F_\rho^{1/2}$ (MeV)	m_{ρ_2} (MeV)
2	625	345.529	1,878.13
4	549.45	345.256	1,874.12
6	531	344.714	1,868.98
8	530	345.445	1,872.08
10	524	345.153	1,869.36
15	518	344.961	1,867.07
20	515	344.848	1,865.81

Table 1: Values of r and z_0 which produce $F_\rho^{1/2} \approx 345$ MeV, and the corresponding second ρ meson masses

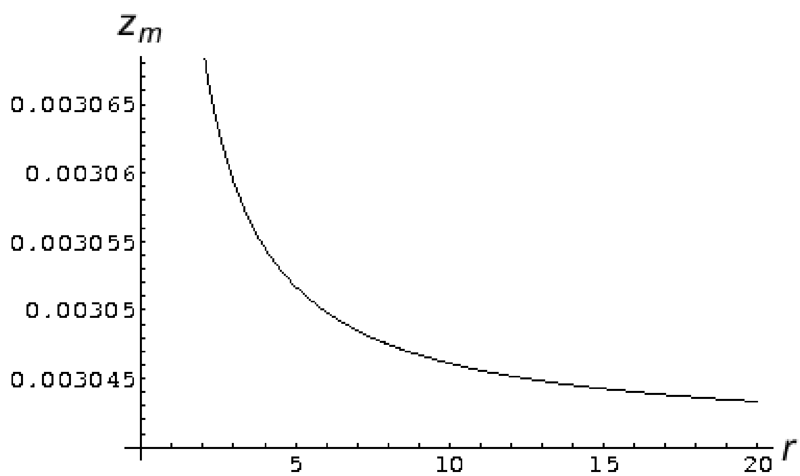


Figure 4: z_m as a function of r for $z_0 = 1/400$ MeV^{-1}

5 Conclusions

Here, we have verified some of the predictions of AdS/QCD, as well as outlined a method of altering the model in the hopes of improving its predictions. The values for the ρ meson masses and for the ρ decay constant that were found matched previous results. We also used our altered model in order to determine new values for the ρ meson mass, so that we may determine how our alterations affect the model's predictions. In fact, our alterations cause this model to produce less accurate predictions than the original model.

6 Further Work

There are many observables which may be calculated using the model described here, other than those presented in this paper. It will be worthwhile to calculate more observables using this model in future research, in order to get a better idea of how our new model affects the predictions of AdS/QCD.

The method described here of altering the geometry of AdS is not an entirely rigorous method of accounting for running of the coupling and is only employed for its simplicity. As stated earlier, a more rigorous method involves allowing the geometry to vary continuously as a function of the extra dimension. This sort of model should be worthwhile pursuing in future research.

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