Computer Modeling of the Δ(1232) Particle

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science
Physics from the College of William and Mary in Virginia,

by

Piers Horvath.

Accepted for: BS in Physics

Advisor: M. Sher

W. J. Kossler

Williamsburg, Virginia
May 2008

Abstract

This report will examine the process of modeling the Δ(1232) resonance in a confined lattice in Maple. This project develops a basic computational model for the decay of the Δ(1232) particle and provides graphs showing the avoided level crossing present in the decay pattern. As an introduction to the Δ(1232) particle, this project also examines a simplified model of ρ meson scattering.

Acknowledgements

I would first like to thank my advisor, Professor Sher, for taking on this project after my former advisor left the College of William and Mary. I would also like to thank Professor Kossler for his guidelines with regards to formatting this thesis. Finally, I would like to thank my friends and family for the support they have provided me while I have been occupied with this project.

Contents

1 Introduction

1.1 Overview

It is known that the Δ(1232) is the first excited state of a nucleon and that this particle almost exclusively decays into a nucleon and a pion.[1] Simulations of this decay process have been made, but the best simulations yet have only been able to simulate down to pion masses of 300 MeV, while physical pion masses are more in the range of 140 MeV. An additional problem with the current simulations is that for pion masses in the range of 300 MeV, the Δ particle is stable, but for physical pion masses, the Δ particle will decay. Recently, a new simulation approach has been attempted whereby the Δ(1232) particle has been studied in the confines of a finite lattice with the aim of finding the resonance parameters for the Δ(1232) decay, m_Δ and $g_{\pi N\Delta}$.[2] This project attempts to use these parameters to plot a graph of the energy levels of the Δ(1232) in a lattice.

2 The ρ Meson

2.1 Introductory Study of Resonance Scattering

In order to examine the Δ(1232), techniques needed to be developed for graphically plotting energy levels and examining avoided level crossings. Due to the complexity of the Δ(1232) interactions, the decision was made to begin with a simplified decay model and graph that as a prelude to the Δ(1232) modeling. An initial attempt has therefore been made to match results found in Gottleib and Rummukainen for the resonance scattering of the ρ meson. Gottleib and Rummukainen studied a simplified scattering model with L=1, and examined this process in both zero and non-zero total momentum cases.[3] This was done in order to expand the theoretical model developed by Lüscher that could only deal with P=0 total momentum.

The initial goal was to reproduce Gottleib and Rummukainen's results with no ρφφ interaction (φ used for the pions to avoid confusion with total momentum P and the ρ meson), and thus no avoided level crossings between the 2φ and ρ energy levels. This was done for both P=0 total momentum as well as P=2π/L total momentum. The first set of levels was plotted with P=0 total momentum according to the formula:

$$W_L = W_{CM} = 2\sqrt{{m_\phi}^2 + {p_\phi}^2}\ , \tag{1}$$

where m_φ = .3/a in this simulation and p_φ = n2π/L, n∈ Z. For the P=2π/L case, the equation for the energy level changed to

$$W_{CM} = \sqrt{W_L^{\,2} - P^2} = \sqrt{\sqrt{p_1^{\,2} + m_\phi^{\,2}} + \sqrt{p_2^{\,2} + m_\phi^{\,2}} - \left(\frac{2\pi}{L}\right)^2} \quad (2)$$

where p_1 and p_2 are the momenta of the individual φ particles given by

$$p_2 = -p_1 + \frac{2\pi}{L} \quad (3)$$

with p_1 still defined by

$$p_1 = \frac{n2\pi}{L}, n \in Z \quad (4)$$

as before. This resulted in the following graph of W_{CM} as a function of lattice size being plotted:

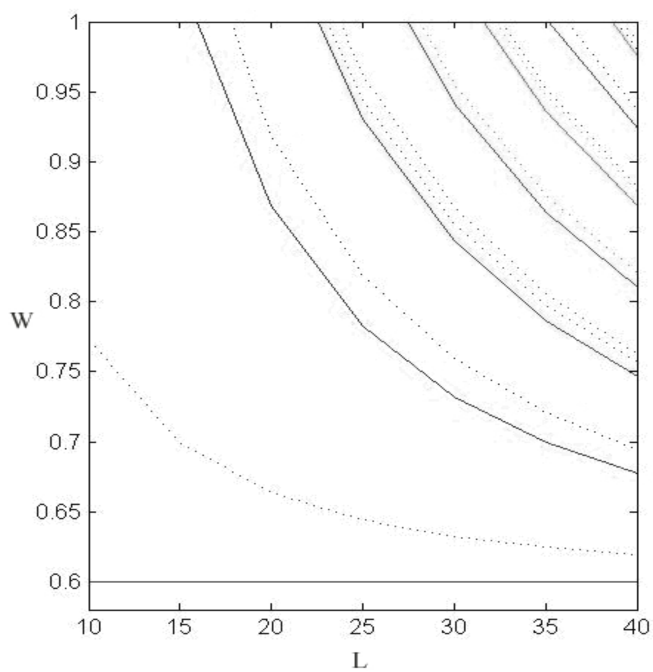


Figure 1: Center of mass energy levels of a system of two non-interacting particles as functions of system size L. Solid lines correspond to total momentum P=0 and dashed lines correspond to P=2π/L.[4]

This project was thus successfully able to replicate the results achieved by Gottleib and Rummukainen for the non-interacting case of φ particles in a lattice.

2.2 Introductory Study of Perturbation Theory

The next goal of this project was the replication of the results achieved by Gottleib and Rummukainen with a small ρφφ interaction. The first decision to be made was how to simulate the interaction between the particles. Gottleib and Rummukainen decided to use a Monte Carlo simulation to calculate the effect, however I decided to use perturbation theory to find the first order effects of interaction between the two lowest energy levels of the ρφφ system. The appropriate formula for this method is

$$E_{\pm} = \frac{1}{2}\left[(W_{bb} + W_{aa}) \pm \sqrt{(W_{aa} - W_{bb})^2 + 4\left|W_{ab}\right|^2} \right] \tag{5}$$

where W_{aa} and W_{bb} are the energy levels of the two interacting levels when calculated independently by formula (2) and W_{ab} is the interaction term that derives from

$$W_{ab} = -2\left(\frac{\hbar^2}{2m_{\phi}} \frac{1}{L^2} \right) \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) t_l(p) \tag{6}$$

For the simulation run by Gottleib and Rummukainen, they simplified the problem to include only the l=0 term, although the physical solution would include the l=1 term. The coefficients P_l are the standard Legendre polynomials and the coefficients $t_l(p)$ are given by the formula

$$t_l(p) = \frac{1}{2i}(e^{i2\delta_l} - 1) \tag{7}$$

with δ_0 given by

$$\delta_0 = \delta_r + \delta_s \tag{8}$$

$$\delta_r = -\lambda_R \frac{p}{16\pi W} + \frac{{g_R}^2}{32\pi} \frac{1}{Wp} \log \frac{4p^2 + {m_\rho}^2}{{m_\rho}^2} \tag{9}$$

$$\tan(\delta_s) = -\frac{{g_R}^2}{16\pi} \frac{p}{W} \frac{1}{W^2 - {m_\rho}^2} \tag{10}$$

The parameters λ_R, g_R, m_φ and m_ρ were determined by Gottleib and Rummukainen through the use of their Monte Carlo simulation. For my first attempt, I attempted to match their results for the condition of small but non-zero coupling. The parameters for this case are: λ_R=36.8, g_R=0.598, m_φ=0.1996 and m_ρ=.5306.

The first test of my simulation is to see whether I observe the same phase shift as Gottleib and Rummukainen. Using formula (8), I was able to create the following graph for this case of small interaction:

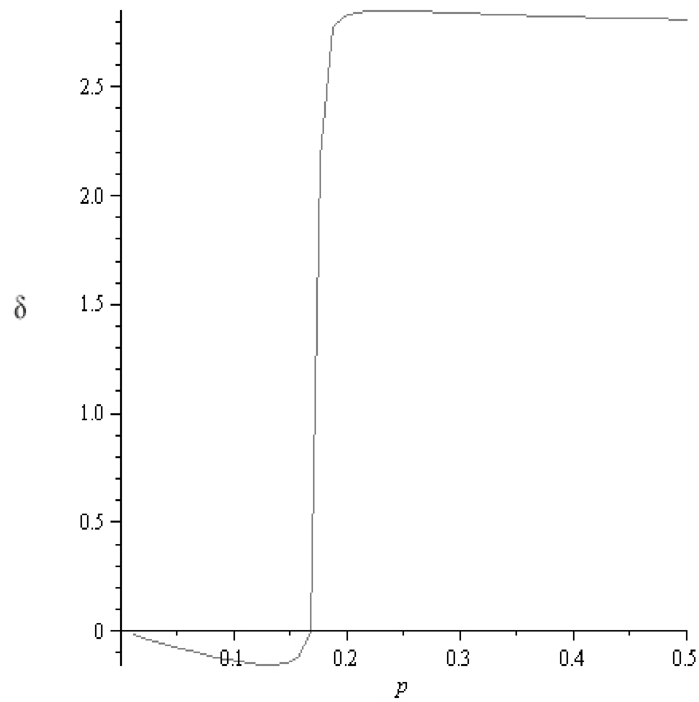


Figure 2: Phase shift versus momentum graph for the case of small interaction.

This graph matches the results of Gottleib and Rummukainen for the phase shift. The next task was to plot the interaction term W_{ab} for $p=2*\pi/L$:

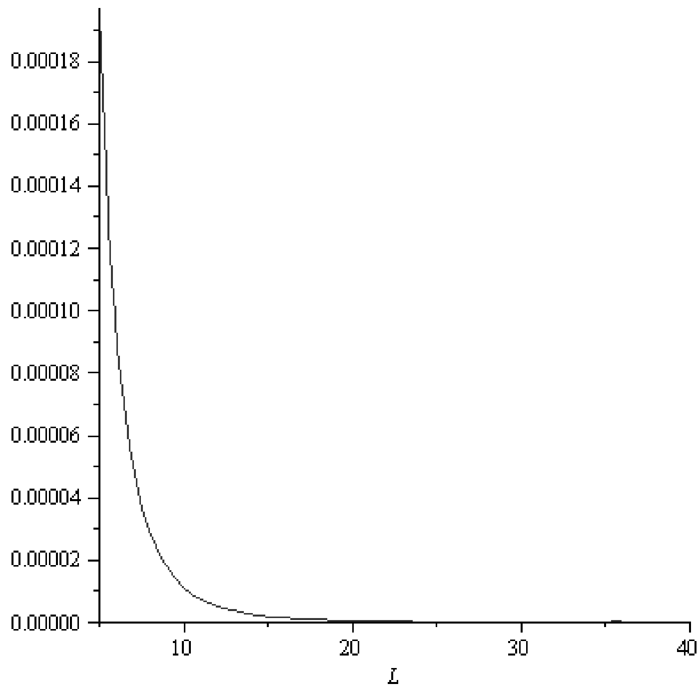


Figure 3: W_{ab} versus L for p=2*π/L.

I next plotted the two discrete energy levels:

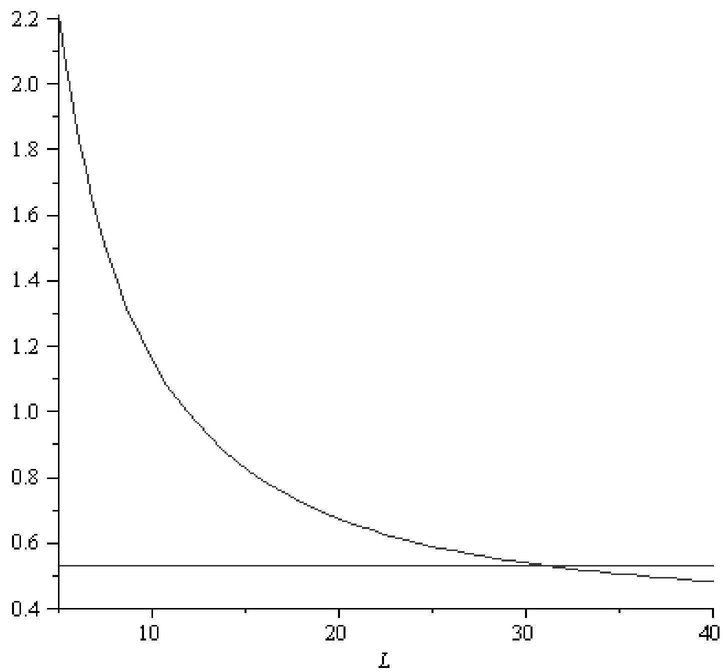


Figure 4: $E_{\pm}$ for p=2*π/L.

When examined more closely, one can see the avoided level crossing present around L=31:

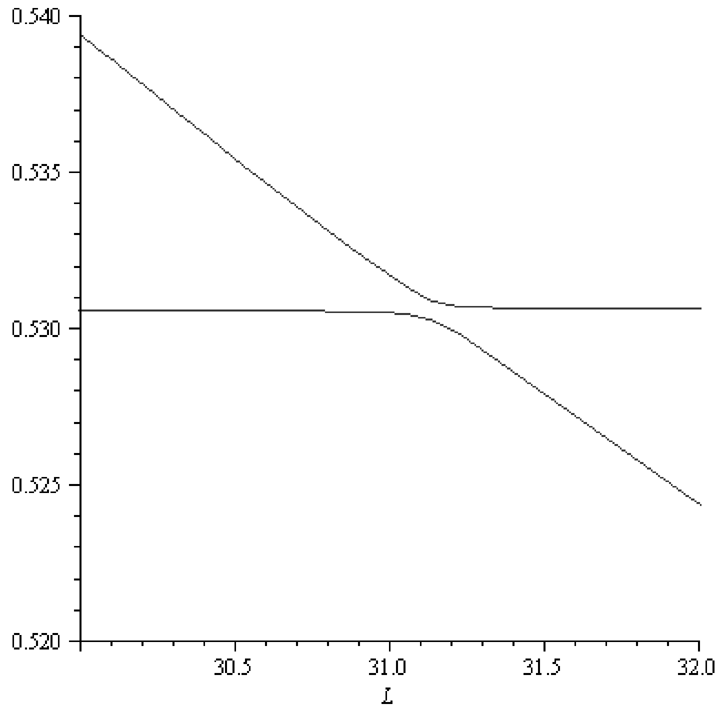


Figure 4: E± for p=2*π/L detail graph.

3 Perturbation and the Δ(1232) Particle

3.1 Introduction to the Δ(1232)

The initial attempt to plot the energy levels of the Δ(1232) particle proceeded along much the same lines as the earlier plot of the ρ meson. The decision was made to first plot the parameterized decay of the Δ(1232) and proceed to the general (in effect, the reverse order of Bernard et al.). This simplified parameterization also uses perturbation theory to plot the decay, but instead of two types of particles as in Gottleib and Rummukainen, there are now three types of particles to consider (the Δ(1232), the pion and the nucleon). Although the parameterized problem is simplified a great deal from the general solution, the same basic equation governs both decay models (equation 20 in Bernard et al.):

$$M_{\Delta}^{2}(1+E_{2,L}(-\omega^{2}))^{2}+\omega^{2}(1-E_{1,L}(-\omega^{2}))^{2}=0 \quad (11)$$ [1]

[1] Throughout this section, the equations will generally be presented with the formatting used when entered into Maple. All of the substitutions made are cosmetic (for example using E instead of Σ for the energy levels to avoid confusion with the sums that enter the equations in the non-parameterized case) and do not affect the final result.

Bernard et al. outline the process whereby this equation can be simplified for the parameterized case. The result of these simplifications is the following equation:

$$E_{1,2}(L) = \frac{1}{2}\left(M_{\Delta} + E_1 \pm \sqrt{(M_{\Delta} - E_1)^2 + 4g(L)}\right) \tag{12}$$

Upon comparison with Eq. (5) it can be seen that this equation represents perturbation theory applied to the Δ(1232) decay with g(L) as the interaction term squared. In order to solve this equation, the following equation definitions are needed:

$$g(L) = \left(\frac{g_{\Pi N\Delta}^2}{F^2}\right)\frac{1}{16(E_1)^3}\left[(E_1 + M_N)^2 - M_{\Pi}^2\right]\lambda\left((E_1)\ , M_N^2, M_{\Pi}^2\right)\frac{1}{L^3 E_{1,N} E_{1,\Pi}} \tag{13}$$

where

$$\lambda\left((E_1)^2, M_N^2, M_{\Pi}^2\right) = (E_1)^4 + M_N^4 + M_{\Pi}^4 - 2(E_1)^2 M_N^2 - 2M_N^2 M_{\Pi}^2 + 2M_{\Pi}^2 (E_1)^2 \tag{14}$$

and

$$E_1 = E_{1,N} + E_{1,\Pi} \tag{15}$$

with those terms defined as

$$E_{1,N} = \sqrt{M_N^2 + (\tfrac{2\pi}{L})^2} \tag{16}$$

and

$$E_{1,\Pi} = \sqrt{M_{\Pi}^2 + (\tfrac{2\pi}{L})^2} \tag{17}$$

M_{Δ}, M_N and M_{Π} are the masses of the Δ(1232), the nucleon and pion respectively, $g_{\Pi N\Delta}$ is the ΠNΔ coupling constant ($g_{\Pi N\Delta}$=1.2) and F is the pion decay constant, which is known to be 92.4 MeV.

3.2 Plotting the Parameterized Δ(1232)

An initial attempt was made to match the results of Bernard et al. using this parameterized system, but the resulting graphs did not correspond to Bernard's results. After review, it was determined that a parameterization constant *k* needed to be introduced into the masses of the particles so that the resulting graphs would match Bernard's results. This was introduced into the modeling as M_N=$M_{N,0}$*k – in essence reducing the values of the masses as put into the model. The value of *k* was experimentally determined to be equal to 2*10^-6 for best results. This resulted in the following graph being plotted:

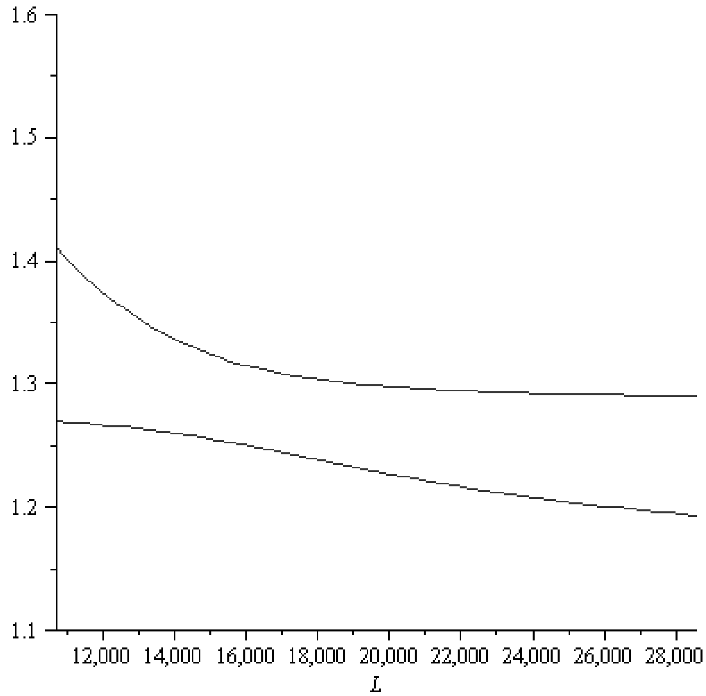


Figure 5: Plot of E1/Mn and E2/Mn for M_{π}=140MeV and $g_{\pi N\Delta}$=1.2. This graph corresponds to the left graph in Figure 7 of Bernard et al: the differing x-axis values represent the same range in both graphs.

This attempt was successful at duplicating Bernard et al. in the construction of a standard graph of a simplified model of the Δ(1232) particle. The next step was to attempt to duplicate the results found for differing parameters that Bernard tested. These changes include altering the ΠNΔ coupling constant to one third of its normal value and increasing the pion mass to 200MeV. The first change made was to increase the pion mass to 200MeV and examine the resulting graph to see if the changes corresponded with results found by Bernard et al.

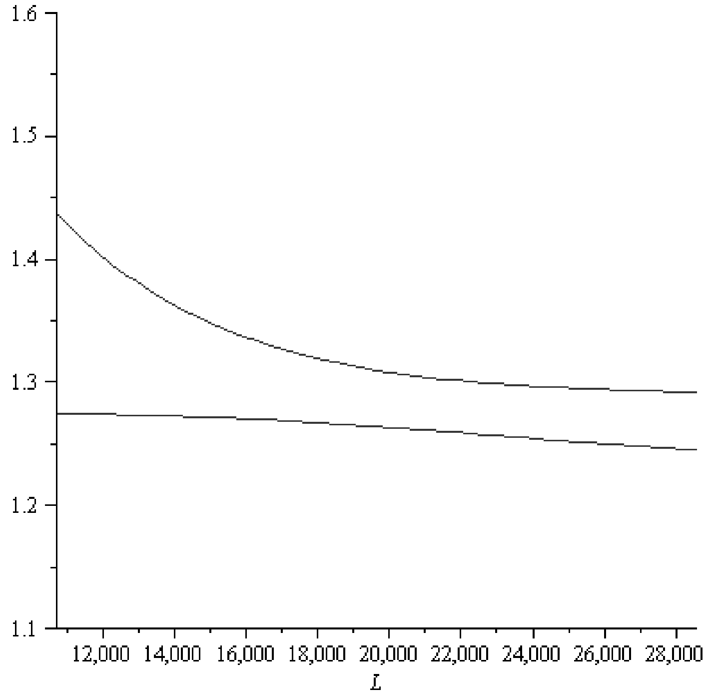


Figure 6: E1/Mn and E2/Mn for M_{Π}=200MeV and $g_{\pi N\Delta}$=1.2.

Unfortunately, the results obtained when increasing the pion mass did not match the results obtained by Bernard et al. The lower energy level was found to have a slightly negative slope, whereas Bernard found a slightly positive slope for the same energy level. For the higher energy level, the Bernard graph started at about 1.6 over this domain, whereas the plotted graph began at about 1.45. This discrepancy is perhaps due to the inexact nature of the parameterization coefficient *k* that was introduced and the differences could perhaps be minimized with further testing of *k* to determine a more exact match.

When reducing the ΠNΔ coupling constant to $g_{\Pi N\Delta}/3$ as Bernard et al. did, a similar pattern emerges; the graph with M_{Π}=140 MeV seems to match the one found by Bernard, but when M_{Π}=200 MeV, the graphs do not correspond.

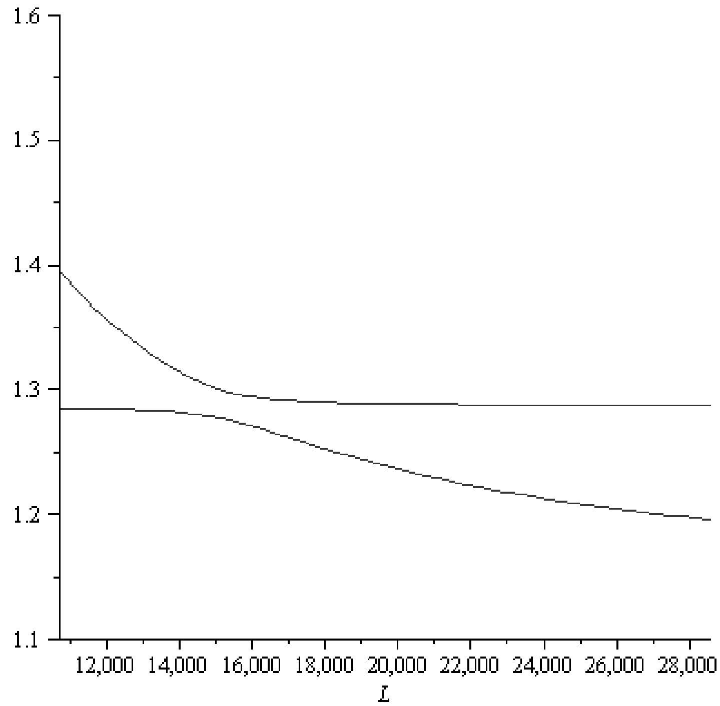


Figure 7: E1/Mn and E2/Mn for M_{Π}=140MeV and $g_{\pi N\Delta}$= $g_{\pi N\Delta}$/3.

The graph of M_{Π}=140MeV and $g_{\pi N\Delta}$= $g_{\pi N\Delta}$/3 does show the same characteristics as Bernard et al. found, namely the avoided level crossing in the vicinity of E/M_n=1.3 as well as the asymptote of E2 at approximately 1.3.

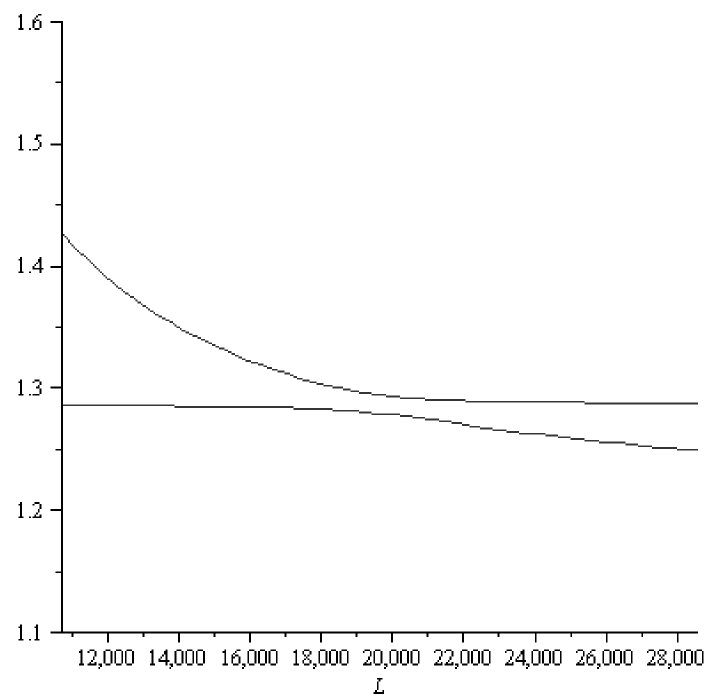


Figure 8: E1/Mn and E2/Mn for M_{Π}=200MeV and $g_{\pi N\Delta}$= $g_{\pi N\Delta}$/3.

The graph of M_{Π}=200MeV and $g_{\pi N\Delta}$= $g_{\pi N\Delta}/3$ shows significant differences from Bernard et al. in the range of the energy levels as well as in the strength of the avoided level crossing. In Bernard et al. the 200MeV pion resulted in an almost stable Δ particle, but in the graphs presented here the Δ(1232) is not stable when the pion mass is 200MeV.

In an attempt to understand how the parameterization method outlined above differed from the general solution (also outlined in Bernard et al.), an attempt was made to match Figure 6 from Bernard.

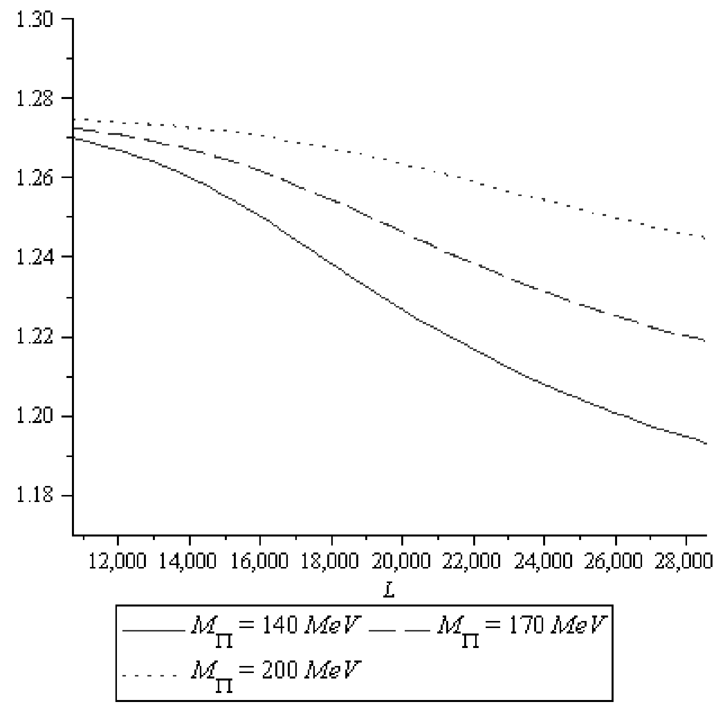


Figure 9: The lowest energy level (E1/Mn) plotted for different values of M_{π}.

An examination of the graph reveals that the parameterized solution does not match the general solution when the value of M_{π} is manipulated. This creates a field for further analysis to determine whether an expression can be generated for the parameterization variable *k* that will enable the parameterized solutions to match the general results.

3.3 The Non-Parameterized Δ(1232)

Bernard et al. also outline the equations whereby the Δ(1232) may be represented more generally. Attempts were made to input the equations generated by Bernard into Maple, but the program was unable to graphically represent the general case (see Appendix 1 for full Maple input). Like the parameterized solution, the general case is governed by Eq. 11:

$$M_{\Delta}{}^{2}(1+E_{2,L}(-\omega^{2}))^{2}+\omega^{2}(1-E_{1,L}(-\omega^{2}))^{2}=0 \tag{11}$$

For the general case, however, the resulting variable definitions differ significantly from the parameterized case:[2]

$$E_{\mathrm{i,L}} = E_{\mathrm{i,N,L}} + E_{\mathrm{i,\Delta,L}}, i = 1,2 \tag{12}$$

with

$$E_{1,N,L} = \frac{g_{\Pi,N,\Delta}^{2}}{F^{2}}\left(W_{2,N} - W_{3,N}\right) \tag{13}$$

$$E_{2,N,L} = \frac{g_{\Pi,N,\Delta}^{2}}{F^{2}}\left(\frac{M_N}{M_\Delta}\right)W_{2,N} \tag{14}$$

and

$$E_{1,\Delta,L} = \frac{5\,g_1^2}{12\,F^2}\left(-T_{0,\Pi} + \left(M_\Delta^2+\omega^2\right)W_{0,\Delta} + \left(M_\Delta^2-\omega^2\right)W_{1,\Delta} - \frac{2}{3\,M_\Delta^2}\left(3\,M_\Delta^2 + \omega^2\right)W_{2,\Delta} - \frac{2}{3\,M_\Delta^2}\left(M_\Delta^2-\omega^2\right)W_{3,\Delta}\right) \tag{15}$$

$$E_{2,\Delta,L} = \frac{5\,g_1^2}{12\,F^2}\left(-T_{0,\Pi} + \left(M_\Delta^2+\omega^2\right)W_{0,\Delta} + 2\,\omega^2\cdot W_{1,\Delta} - \frac{4}{3}W_{2,\Delta} - \frac{4\,\omega^2}{3\,M_\Delta^2}W_{3,\Delta}\right) \tag{16}$$

These equations in turn call upon the following definitions:

$$W_{0,N} = H_1 + H_2 + H_3 \tag{17}$$

$$W_{2,N} = \frac{1}{12\,\omega^2}\left(\left(\omega^4 + M_N^4 + M_\Pi^4 + 2\,\omega^2\cdot M_N^2 - 2\,M_N^2\cdot M_\Pi^2 - 2\,M_\Pi^2\cdot\omega^2\right)\cdot W_{0,N} - \left(\omega^2 + M_N^2 - M_\Pi^2\right)\cdot T_{0,\Pi} - \left(\omega^2 - M_N^2 + M_\Pi^2\right)\cdot T_{0,N}\right) \tag{18}$$

$$W_{3,N} = \frac{\omega^2 + M_N^2 - M_\Pi^2}{2\,\omega^2}W_{2,N} - \frac{1}{6\,\omega^2}\left(M_\Pi^2\cdot T_{0,\Pi} - T_{2,\Pi} - M_N^2\cdot T_{0,N} + T_{2,N}\right) \tag{19}$$

that call upon another subset of equations defined as:

$$H_1 = \left(\omega^2-\mu^2\right)\frac{1}{L^3}\sum_{n=-\infty}^{\infty}\left(\frac{E_N+E_\Pi}{2\cdot E_N\cdot E_\Pi}\cdot\frac{1}{\omega^2+\left(E_N+E_\Pi\right)^2}\cdot\frac{1}{\left(\mu^2+\left(E_N+E_\Pi\right)^2\right)^2}\right) \tag{20}$$

$$H_2 = \frac{1}{8\cdot\pi^2}\int_0^1\left(\sum_{j\neq 0}\left(K_0\left(L\cdot j\cdot \mathrm{sqrt}\left(g_N\right)\right) - \left(\omega^2-\mu^2\right)\frac{x(1-x)L\cdot j}{2\cdot\mathrm{sqrt}\left(g_N\right)}\cdot K_1\left(L\cdot j\cdot\mathrm{sqrt}\left(g_N\right)\right)\right)\right)\mathrm{d}x \tag{21}$$

---

[2] As the relations between the various equations can be confusing, I have included an equation map in Appendix 2 as a guide to the relation between equations for the general case.

$$H_3 = \frac{-B_\omega}{32\cdot\pi^2\cdot\omega^2}\left(\ln\left(\frac{\left(-\omega^2 + M_N^2 - M_\Pi^2 + B_\omega\right)}{\left(-\omega^2 + M_N^2 - M_\Pi^2 - B_\omega\right)}\right) + \ln\left(\frac{\left(-\omega^2 - M_N^2 - M_\Pi^2 + B_\omega\right)}{\left(-\omega^2 - M_N^2 - M_\Pi^2 - B_\omega\right)}\right)\right)$$
$$+ \frac{B_\mu}{16\cdot\pi^2\cdot\mu^2}\left(\arctan\left(\frac{-\mu^2 + M_N^2 - M_\Pi^2}{B_\mu}\right) + \arctan\left(\frac{-\mu^2 - M_N^2 - M_\Pi^2}{B_\mu}\right)\right)$$
$$- \frac{\omega^2 - \mu^2}{16\cdot\pi^2\cdot\mu^2}\left(1 + \frac{(\omega^2 - \mu^2)\left(M_N^2 - M_\Pi^2\right)}{B_\mu}\ln\left(\frac{M_N^2}{M_\Pi^2}\right)\right.$$
$$- \frac{\mu^2\left(M_N^2 + M_\Pi^2\right) + \left(M_N^2 - M_\Pi^2\right)^2}{\mu^2 B_\mu}\cdot\left(\arctan\left(\frac{-\mu^2 + M_N^2 - M_\Pi^2}{B_\mu}\right)\right.$$
$$\left.\left.\left. + \arctan\left(\frac{-\mu^2 - M_N^2 - M_\Pi^2}{B_\mu}\right)\right)\right)\right) \quad (22)$$

where $K_v(z)$ is the modified Bessel function.
Furthermore, these last equations call on the following set of equations:

$$T_{0,N} = \frac{M_N^2}{4\cdot\pi^2}\left(\sum_{j\neq 0}\left(\frac{K_1\left(M_N\cdot L\cdot j\right)}{M_N\cdot L\cdot j}\right)\right) \quad (23)$$

$$T_{0,\Pi} = \frac{M_\Pi^2}{4\cdot\pi^2}\left(\sum_{j\neq 0}\left(\frac{K_1\left(M_\Pi\cdot L\cdot j\right)}{M_\Pi\cdot L\cdot j}\right)\right) \quad (24)$$

$$T_{2,N} = \frac{-M_N^4}{4\cdot\pi^2}\left(\sum_{j\neq 0}\left(\frac{K_2\left(M_N\cdot L\cdot j\right)}{\left(M_N\cdot L\cdot j\right)^2}\right)\right) \quad (25)$$

$$T_{2,\Pi} = \frac{-M_N^4}{4\cdot\pi^2}\left(\sum_{j\neq 0}\left(\frac{K_2\left(M_\Pi\cdot L\cdot j\right)}{\left(M_\Pi\cdot L\cdot j\right)^2}\right)\right) \quad (26)$$

The above equations also need the following equation definitions:

$$g_N = (1-x)M_\Pi^2 + x\cdot M_N^2 + x(1-x)\mu^2 \quad (27)$$

$$B_\mu = \mathrm{sqrt}\left(\mu^4 + M_N^4 + M_\Pi^4 + 2\,\mu^2\cdot M_N^2 - 2\,M_N^2\cdot M_\Pi^2 - 2\,M_\Pi^2\cdot\mu^2\right) \quad (28)$$

$$B_\omega = \mathrm{sqrt}\left(\omega^4 + M_N^4 + M_\Pi^4 + 2\,\omega^2\cdot M_N^2 - 2\,M_N^2\cdot M_\Pi^2 - 2\,M_\Pi^2\cdot\omega^2\right) \quad (29)$$

$$E_\Pi = \mathrm{sqrt}\left(M_\Pi^2 + k_n^2\right) \quad (30)$$

$$E_N = \mathrm{sqrt}\left(M_N^2 + k_n^2\right) \quad (31)$$

Note that for the general case, k_n is the lattice momenta 2πn/L. The above equations define the nucleon equations fully, but we need still more equations (as provided by Bernard et al.) to complete the Δ equations:

$$W_{0,\Delta} = \frac{1}{8\cdot\pi^2}\int_0^1\left(\sum_{j\neq 0}\left(K_0\left(L\cdot j\cdot\mathrm{sqrt}\left(g_\Delta\right)\right)\right)\right)dx \quad (32)$$

$$W_{1,\Delta} = \frac{\omega^2 + M_\Delta^2 - M_\Pi^2}{2\,\omega^2}W_{0,\Delta} - \frac{T_{0,\Pi}}{2\,\omega^2} + \frac{T_{0,\Delta}}{2\,\omega^2} \quad (33)$$

$$W_{2,\Delta} = \frac{1}{12\,\omega^2}\Big(\big(\omega^4 + M_\Delta^4 + M_\Pi^4 + 2\,\omega^2\cdot M_\Delta^2 - 2\,M_\Delta^2\cdot M_\Pi^2 - 2\,M_\Pi^2\cdot\omega^2\big)\cdot W_{0,\Delta} - \big(\omega^2 + M_\Delta^2 - M_\Pi^2\big)\cdot T_{0,\Pi} - \big(\omega^2 - M_\Delta^2 + M_\Pi^2\big)\cdot T_{0,\Delta}\Big) \tag{34}$$

$$W_{3,\Delta} = \frac{\omega^2 + M_\Delta^2 - M_\Pi^2}{2\,\omega^2} W_{2,\Delta} - \frac{1}{6\,\omega^2}\big(M_\Pi^2\cdot T_{0,\Pi} - T_{2,\Pi} - M_\Delta^2\cdot T_{0,\Delta} + T_{2,\Delta}\big) \tag{35}$$

These equations again call upon further equations:

$$T_{0,\Delta} = \frac{M_\Delta^2}{4\cdot\pi^2}\left(\sum_{j\neq\infty}\left(\frac{K_1(M_\Delta\cdot L\cdot j)}{M_\Delta\cdot L\cdot j}\right)\right) \tag{36}$$

$$T_{2,\Delta} = \frac{-M_\Delta^4}{4\cdot\pi^2}\left(\sum_{j\neq 0}\left(\frac{K_2(M_\Delta\cdot L\cdot j)}{(M_\Delta\cdot L\cdot j)^2}\right)\right) \tag{37}$$

$$g_\Delta = (1-x)M_\Pi^2 + x\cdot M_\Delta^2 + x(1-x)\omega^2 \tag{38}$$

The equations above fully define the system, with the masses and coupling constants equal to their values in the parameterization case. One additional coupling constant is needed to define the system: g_1=2.0 (defined by Bernard et al.).

After putting the above equations into Maple, efforts were made to solve Eq. 11 and produce a graph of the energy levels for the Δ(1232) particle. These efforts did not succeed, and the results of Bernard et al. were not able to be duplicated. One potential source for error in these simulations was the definition of certain variables (specifically μ and ω), as these variables were not defined in the Bernard article. Attempts were made to graph the general case for a number of different values of μ and ω but none of these attempts were successful enough to produce a graph.

4 Conclusions

Efforts to replicate the graphs found in Bernard et al. for the energy levels of the Δ(1232) particle were partially successful. For the parameterized case, graphs were obtained that matched the Bernard graphs, although some work could still be done to ensure a more exact match. In particular, the behavior of the energy levels as certain parameters were modified (such as the pion mass or the ΠNΔ coupling constant) did not match Bernard's work and could be improved upon with further testing. As regards the general case of the energy levels of the Δ(1232) particle, no graphs were able to be produced and compared to Bernard's work. The project was a slight success with regards to the general case, however, as the equations were able to be input into Maple in a logical framework that could be developed for further projects.

[1] Pascalutsa et al. Phys. Rept 437 (2007) 125-232.
[2] Bernard et al. (2007) [arXiv:hep-lat/070201v1] 1-20.
[3] Gottleib and Rummukainen (1995) [hep-lat/9509088] 1-4.
[4] Rummukainen and Gottleib, Nucl. Phys. B. 450 (1995) 397-436.

Appendix 1: Maple Input

Instantiation of Variables:

M_{Π};

M_N;

M_{Δ};

L;

ω;

μ;

n;

$k_n := \frac{2\cdot \text{Pi}\cdot n}{L}$;

$g_{\Pi, N, \Delta}$;

g_1;

F;

Level 1 Equations (N side):

$g_N := (1-x)M_{\Pi}^2 + x\cdot M_N^2 + x(1-x)\mu^2$;

$B_{\mu} := \text{sqrt}\left(\mu^4 + M_N^4 + M_{\Pi}^4 + 2\,\mu^2\cdot M_N^2 - 2\,M_N^2\cdot M_{\Pi}^2 - 2\,M_{\Pi}^2\cdot\mu^2\right)$;

$B_{\omega} := \text{sqrt}\left(\omega^4 + M_N^4 + M_{\Pi}^4 + 2\,\omega^2\cdot M_N^2 - 2\,M_N^2\cdot M_{\Pi}^2 - 2\,M_{\Pi}^2\cdot\omega^2\right)$;

$E_{\Pi} := \text{sqrt}\left(M_{\Pi}^2 + k_n^2\right)$;

$E_N := \text{sqrt}\left(M_N^2 + k_n^2\right)$;

Level 2 Equations (N side):

$$H_1 := \left(\omega^2 - \mu^2\right)\frac{1}{L^3}sum\left(\frac{E_N + E_{\Pi}}{2\cdot E_N\cdot E_{\Pi}}\cdot\frac{1}{\omega^2 + \left(E_N + E_{\Pi}\right)^2}\cdot\frac{1}{\left(\mu^2 + \left(E_N + E_{\Pi}\right)^2\right)^2}, n = -\infty \,...\, \infty\right);$$

$$H_2 := \frac{1}{8\cdot\pi^2}\int_0^1 \left(\sum_{j=-\infty}^{-1}\left(\text{BesselK}\left(0, L\cdot j\cdot \text{sqrt}(g_N)\right) - \left(\omega^2 - \mu^2\right)\frac{x(1-x)L\cdot j}{2\cdot\text{sqrt}(g_N)}\cdot\text{BesselK}\left(1, \left(L\cdot j\cdot\text{sqrt}(g_N)\right)\right)\right) + \sum_{j=1}^{\infty}\left(\text{BesselK}\left(0, L\cdot j\cdot\text{sqrt}(g_N)\right) - \left(\omega^2 - \mu^2\right)\frac{x(1-x)L\cdot j}{2\cdot\text{sqrt}(g_N)}\cdot\text{BesselK}\left(1, \left(L\cdot j\cdot\text{sqrt}(g_N)\right)\right)\right)\right)\mathrm{d}x;$$

$$H_3 := \frac{-B_\omega}{32\cdot\pi^2\cdot\omega^2}\left(\ln\left(\frac{\left(-\omega^2 + M_N^2 - M_\Pi^2 + B_\omega\right)}{\left(-\omega^2 + M_N^2 - M_\Pi^2 - B_\omega\right)}\right) + \ln\left(\frac{\left(-\omega^2 - M_N^2 - M_\Pi^2 + B_\omega\right)}{\left(-\omega^2 - M_N^2 - M_\Pi^2 - B_\omega\right)}\right)\right) + \frac{B_\mu}{16\cdot\pi^2\cdot\mu^2}\left(\arctan\left(\frac{-\mu^2 + M_N^2 - M_\Pi^2}{B_\mu}\right) + \arctan\left(\frac{-\mu^2 - M_N^2 - M_\Pi^2}{B_\mu}\right)\right) - \frac{\omega^2 - \mu^2}{16\cdot\pi^2\cdot\mu^2}\left(1 + \frac{\left(\omega^2 - \mu^2\right)\left(M_N^2 - M_\Pi^2\right)}{B_\mu}\ln\left(\frac{M_N^2}{M_\Pi^2}\right) - \frac{\mu^2\left(M_N^2 + M_\Pi^2\right) + \left(M_N^2 - M_\Pi^2\right)^2}{\mu^2 B_\mu}\cdot\left(\arctan\left(\frac{-\mu^2 + M_N^2 - M_\Pi^2}{B_\mu}\right) + \arctan\left(\frac{-\mu^2 - M_N^2 - M_\Pi^2}{B_\mu}\right)\right)\right);$$

$$T_{0,N} := \frac{M_N^2}{4\cdot\pi^2}\left(\sum_{j=-\infty}^{-1}\left(\frac{\text{BesselK}\left(1, M_N\cdot L\cdot j\right)}{M_N\cdot L\cdot j}\right) + \sum_{j=1}^{\infty}\left(\frac{\text{BesselK}\left(1, M_N\cdot L\cdot j\right)}{M_N\cdot L\cdot j}\right)\right);$$

$$T_{0,\Pi} := \frac{M_\Pi^2}{4\cdot\pi^2}\left(\sum_{j=-\infty}^{-1}\left(\frac{\text{BesselK}\left(1, M_\Pi\cdot L\cdot j\right)}{M_\Pi\cdot L\cdot j}\right) + \sum_{j=1}^{\infty}\left(\frac{\text{BesselK}\left(1, M_\Pi\cdot L\cdot j\right)}{M_\Pi\cdot L\cdot j}\right)\right);$$

$$T_{2,N} := \frac{-M_N^4}{4\cdot\pi^2}\left(\sum_{j=-\infty}^{-1}\left(\frac{\text{BesselK}\left(2, M_N\cdot L\cdot j\right)}{\left(M_N\cdot L\cdot j\right)^2}\right) + \sum_{j=1}^{\infty}\left(\frac{\text{BesselK}\left(2, M_N\cdot L\cdot j\right)}{\left(M_N\cdot L\cdot j\right)^2}\right)\right);$$

$$T_{2,\Pi} := \frac{-M_\Pi^4}{4\cdot\pi^2}\left(\sum_{j=-\infty}^{-1}\left(\frac{\text{BesselK}\left(2, M_\Pi\cdot L\cdot j\right)}{\left(M_\Pi\cdot L\cdot j\right)^2}\right) + \sum_{j=1}^{\infty}\left(\frac{\text{BesselK}\left(2, M_\Pi\cdot L\cdot j\right)}{\left(M_\Pi\cdot L\cdot j\right)^2}\right)\right);$$

Level 3 Equations (N side):

$$W_{0,N} := H_1 + H_2 + H_3;$$

$$W_{2,N} := \frac{1}{12\,\omega^2}\left(\left(\omega^4 + M_N^4 + M_\Pi^4 + 2\,\omega^2\cdot M_N^2 - 2\,M_N^2\cdot M_\Pi^2 - 2\,M_\Pi^2\cdot\omega^2\right)\cdot W_{0,N} - \left(\omega^2 + M_N^2 - M_\Pi^2\right)\cdot T_{0,\Pi} - \left(\omega^2 - M_N^2 + M_\Pi^2\right)\cdot T_{0,N}\right);$$

$$W_{3,N} := \frac{\omega^2 + M_N^2 - M_\Pi^2}{2\,\omega^2}W_{2,N} - \frac{1}{6\,\omega^2}\left(M_\Pi^2\cdot T_{0,\Pi} - T_{2,\Pi} - M_N^2\cdot T_{0,N} + T_{2,N}\right);$$

Level 4 Equations (N side):

$$E_{1,N,L} := \frac{g_{\Pi,N,\Delta}^2}{F^2}\left(W_{2,N} - W_{3,N}\right);$$

$$E_{2,N,L} := \frac{g_{\Pi,N,\Delta}^2}{F^2}\left(\frac{M_N}{M_\Delta}\right)W_{2,N};$$

End of N section.

Level 1 Equations (Δ Side):

$$g_\Delta := (1-x)M_\Pi^2 + x\cdot M_\Delta^2 + x(1-x)\omega^2;$$

Level 2 Equations (Δ Side):

$$T_{0,\Delta} := \frac{M_\Delta^2}{4\cdot\pi^2}\left(\sum_{j=-\infty}^{-1}\left(\frac{\mathrm{BesselK}(1, M_\Delta\cdot L\cdot j)}{M_\Delta\cdot L\cdot j}\right) + \sum_{j=1}^{\infty}\left(\frac{\mathrm{BesselK}(1, M_\Delta\cdot L\cdot j)}{M_\Delta\cdot L\cdot j}\right)\right);$$

$$T_{2,\Delta} := \frac{-M_\Delta^4}{4\cdot\pi^2}\left(\sum_{j=-\infty}^{-1}\left(\frac{\mathrm{BesselK}(2, M_\Delta\cdot L\cdot j)}{(M_\Delta\cdot L\cdot j)^2}\right) + \sum_{j=1}^{\infty}\left(\frac{\mathrm{BesselK}(2, M_\Delta\cdot L\cdot j)}{(M_\Delta\cdot L\cdot j)^2}\right)\right);$$

Level 3 Equations (Δ side):

$$W_{0,\Delta} := \frac{1}{8\cdot\pi^2}\int_0^1\left(\sum_{j=-\infty}^{-1}\left(\mathrm{BesselK}\left(0, L\cdot j\cdot \mathrm{sqrt}(g_\Delta)\right)\right) + \sum_{j=1}^{\infty}\left(\mathrm{BesselK}\left(0, L\cdot j\cdot \mathrm{sqrt}(g_\Delta)\right)\right)\right)\mathrm{d}x;$$

$$W_{1,\Delta} := \frac{\omega^2 + M_\Delta^2 - M_\Pi^2}{2\,\omega^2}W_{0,\Delta} - \frac{T_{0,\Pi}}{2\,\omega^2} + \frac{T_{0,\Delta}}{2\,\omega^2};$$

$$W_{2,\Delta} := \frac{1}{12\,\omega^2}\Big(\Big(\omega^4 + M_\Delta^4 + M_\Pi^4 + 2\,\omega^2\cdot M_\Delta^2 - 2\,M_\Delta^2\cdot M_\Pi^2 - 2\,M_\Pi^2 \cdot\omega^2\Big)\cdot W_{0,\Delta} - \Big(\omega^2 + M_\Delta^2 - M_\Pi^2\Big)\cdot T_{0,\Pi} - \Big(\omega^2 - M_\Delta^2 + M_\Pi^2\Big) \cdot T_{0,\Delta}\Big);$$

$$W_{3,\Delta} := \frac{\omega^2 + M_\Delta^2 - M_\Pi^2}{2\,\omega^2} W_{2,\Delta} - \frac{1}{6\,\omega^2}\Big(M_\Pi^2\cdot T_{0,\Pi} - T_{2,\Pi} - M_\Delta^2 \cdot T_{0,\Delta} + T_{2,\Delta}\Big);$$

Level 4 Equations (Δ Side):

$$E_{1,\Delta,L} := \frac{5\,g_1^2}{12\,F^2}\left(-T_{0,\Pi} + \Big(M_\Delta^2 + \omega^2\Big)W_{0,\Delta} + \Big(M_\Delta^2 - \omega^2\Big)W_{1,\Delta} - \frac{2}{3\,M_\Delta^2}\Big(3\,M_\Delta^2 + \omega^2\Big)W_{2,\Delta} - \frac{2}{3\,M_\Delta^2}\Big(M_\Delta^2 - \omega^2\Big)W_{3,\Delta}\right);$$

$$E_{2,\Delta,L} := \frac{5\,g_1^2}{12\,F^2}\left(-T_{0,\Pi} + \Big(M_\Delta^2 + \omega^2\Big)W_{0,\Delta} + 2\,\omega^2\cdot W_{1,\Delta} - \frac{4}{3}W_{2,\Delta} - \frac{4\,\omega^2}{3\,M_\Delta^2}W_{3,\Delta}\right);$$

End of Δ section.

Level 5 Equations:

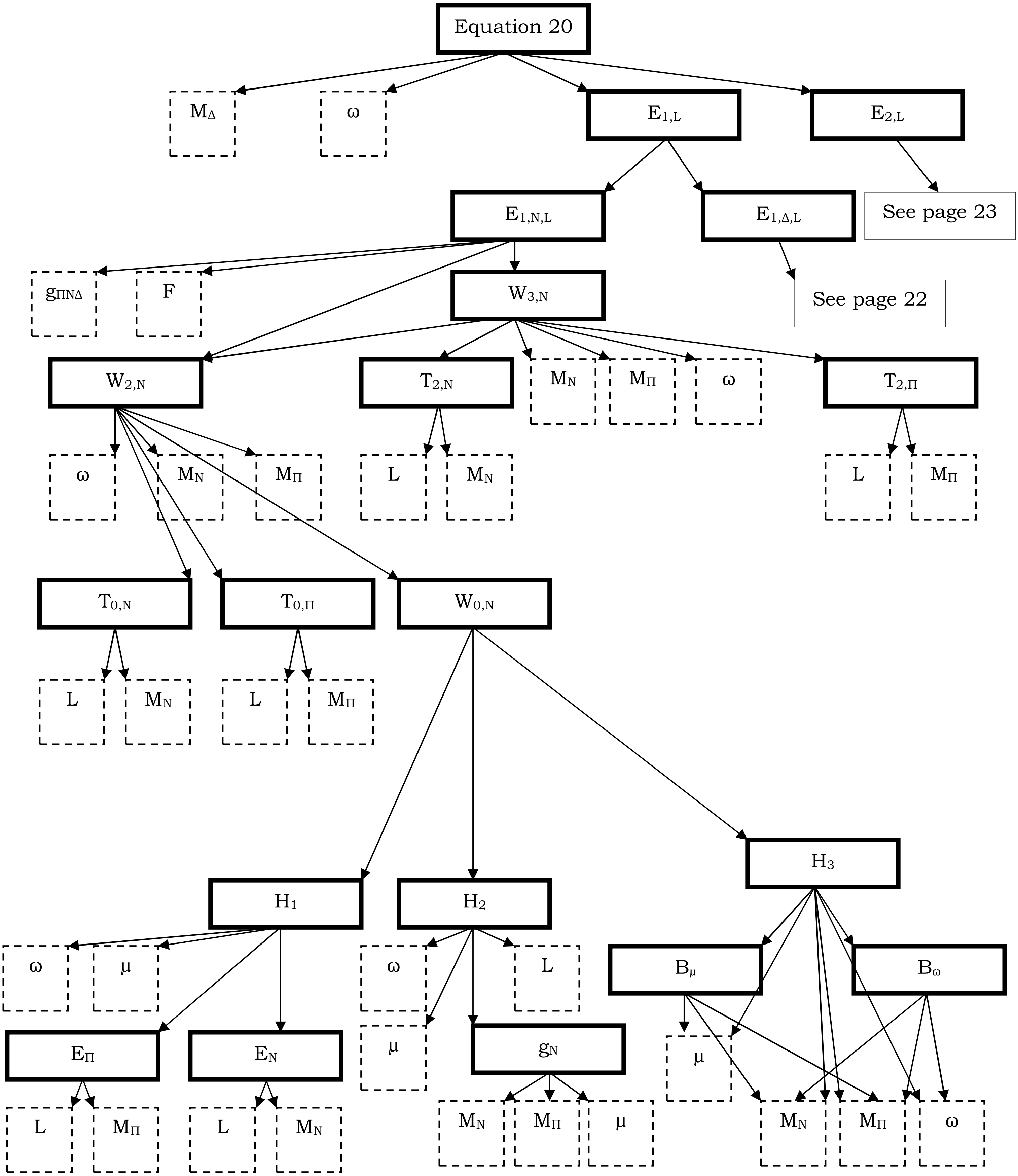
$$E_{1,L} := E_{1,N,L} + E_{1,\Delta,L};$$

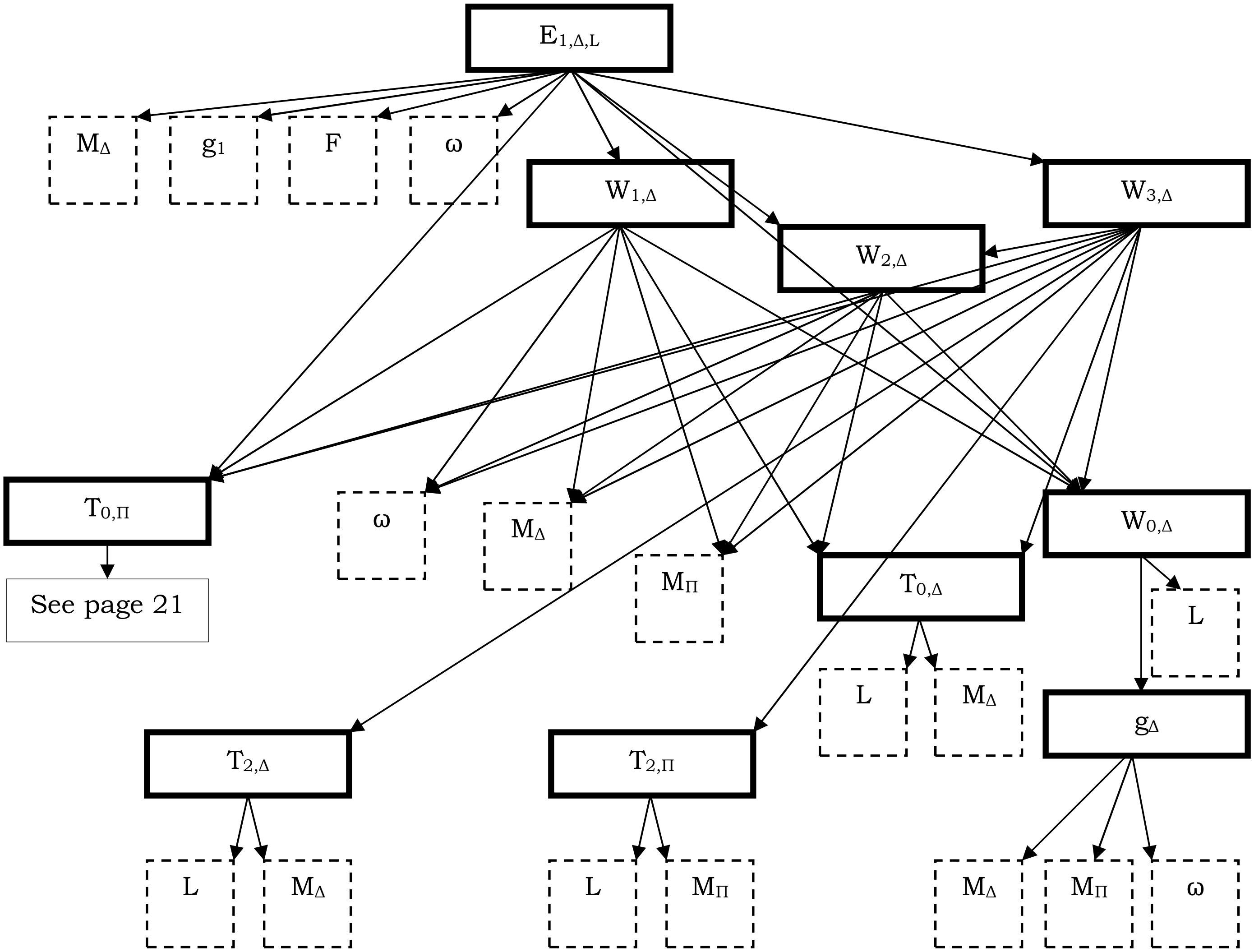
$$E_{2,L} := E_{2,N,L} + E_{2,\Delta,L};$$

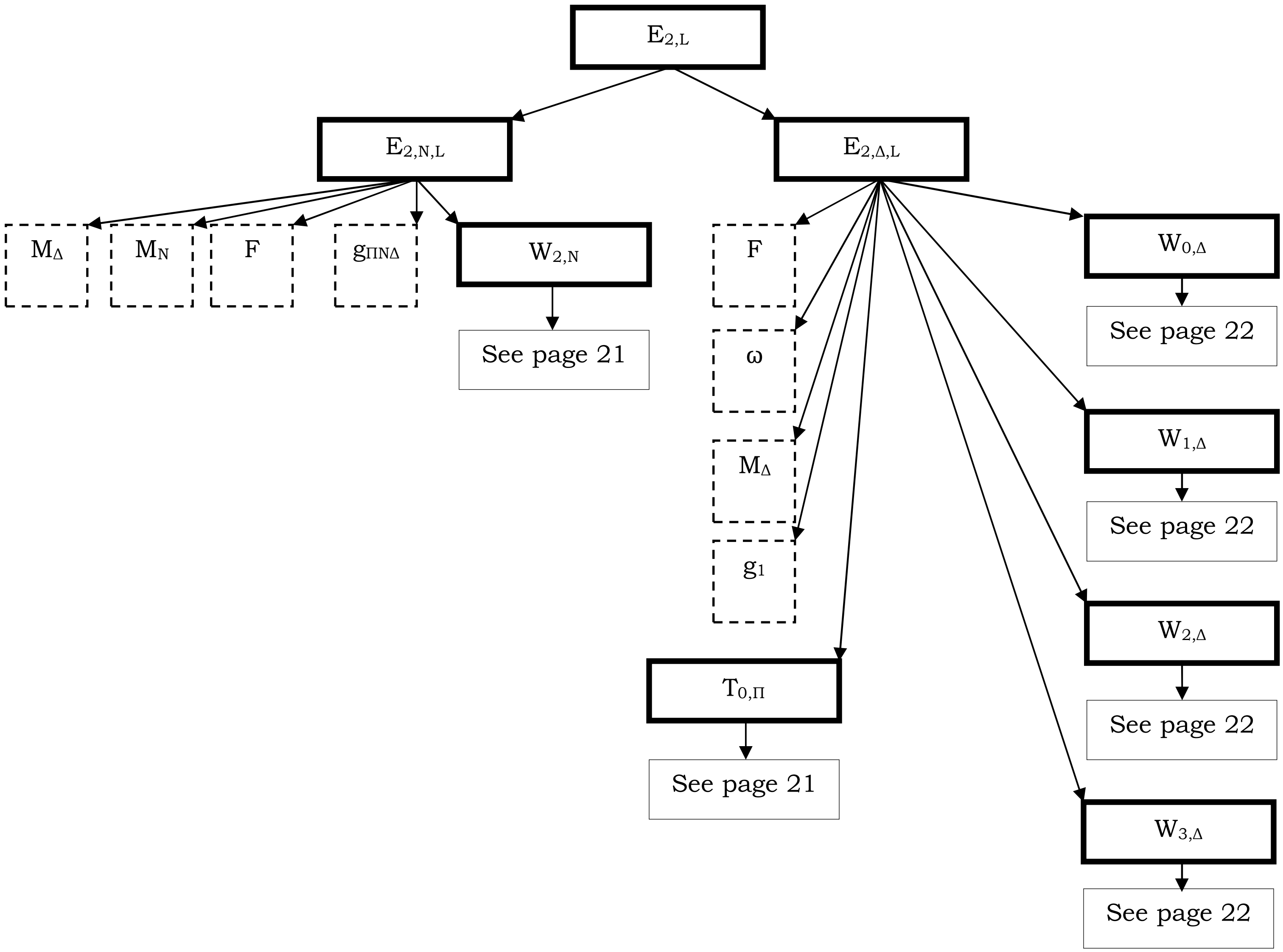
Equation 20:

$$M_\Delta^2\Big(1 + E_{2,L}\Big)^2 + \omega^2\Big(1 - E_{1,L}\Big)^2 = 0;$$

Appendix 2: Equation Map




$E_{1,\Delta,L}$
M_Δ
g_1
F
ω
$W_{1,\Delta}$
$W_{3,\Delta}$
$W_{2,\Delta}$
$T_{0,\Pi}$
ω
M_Δ
$W_{0,\Delta}$
See page 21
M_Π
$T_{0,\Delta}$
L
L
M_Δ
g_Δ
$T_{2,\Delta}$
$T_{2,\Pi}$
L
M_Δ
L
M_Π
M_Δ
M_Π
ω


E2,L
E2,N,L
E2,Δ,L
MΔ
MN
F
gΠNΔ
W2,N
See page 21
F
ω
MΔ
g1
T0,Π
See page 21
W0,Δ
See page 22
W1,Δ
See page 22
W2,Δ
See page 22
W3,Δ
See page 22