Warp Duality in Braneworlds

Andrew B. McGowan

September 14, 2007

Abstract

In recent years there have emerged numerous models of spacetime that include extra dimensions. In particular there have been a variety of 'brane-worlds,' scenarios in which we live on a 3+1 dimensional subspace of a higher dimensional spacetime. More specifically, in the Randall-Sundrum 2 braneworld there are five dimensions, but at lower energies gravity approximately obeys the four-dimensional Einstein's Equations. We investigate the possibility that multiple five dimensional spacetimes could give rise to the same four-dimensional physics. If so, it would be more difficult to experimentally determine the shape of any possible extra dimensions.

1 Introduction

1.1 The History of Extra Dimensions

The history of modern approaches to physical questions employing the use of extra dimensions begins with Gunnar Nordstrom, who in 1914 attempted to unify gravity and electromagnetism with the addition of a fourth spatial dimension [1]. While in some ways Nordstrom was successful, his work came before the discovery of General Relativity, so his understanding of gravity was limited to an approximate theory. Nonetheless, he established an important precedent of looking for answers in extra dimensions.

The next watershed event occurred in 1921, when Theodor Kaluza discovered that the equations of General Relativity with an extra spatial dimension could include Maxwell's Equations [2] [3]. The problem was that Kaluza had to assume that none of the fields in the universe varied over the extra dimension. In 1926, Oscar Klein offered a solution to this quandary with the suggestion that, if the extra dimensions were very small compared to the distance scales we normally observe, then the fields would remain constant over the extra dimension [4]. We now know the resulting theory as 'Kaluza Klein' theory.

Kaluza Klein theory, however, had numerous problems [5], not the least of which was that its prediction for the unit of charge was many times smaller than the charge of the electron, so the idea of extra dimensions largely fell by the wayside for many years until the advent of String Theory in the 1960's.

Initially String Theory was an effort to describe the strong force, but it later became a candidate for a Quantum Theory of Gravity [6].

While initially String Theory appeared to be quite promising, in early formulations it suffered from a number of problems, which physicists slowly resolved through the 70's and 80's. The solutions, however, required at least ten dimensions, so the idea of extra dimensions came back onto the scene [6].

In the 1990's, Horava and Witten [6] (among others) discovered that string theory contained objects called 'p-branes,' which are essentially p-dimensional generalizations of membranes. As it turned out, many p-branes have the property that open strings can end on them, and that the oscillations of these open strings contain modes that resemble the electromagnetic field. This discovery suggested a new way to look at the possibility of extra dimensions: the 'braneworld.' [6]

In braneworld scenarios, we live on a 3+1 dimensional p-brane, which is in turn embedded in some higher dimensional space [6]. It is these braneworld scenarios that are the subject of our investigation. In particular, we are interested in variations on the Randall-Sundrum braneworlds [7] [8], which provide two possibilities of braneworlds that would be consistent with our observed universe. More specifically, we investigate whether there might be dualities in the Randall-Sundrum braneworlds–whether it is possible that different extra-dimensional geometries give rise to the same four-dimensional observed physics. If it is, then it may not be possible to experimentally determine the shape of the extra dimensions. If the Randall-Sundrum scenarios are unique, then they are more likely experimentally verifiable.

1.2 Motivation for Extra Dimensions

There are a number of motivations for exploring extra dimensions. The first is the hierarchy problem for the relative strength of the forces: why is gravity so weak? Essentially, the idea is that extra dimensions may provide a way to dissipate the strength of gravity. Since gravity is a property intrinsic to space itself, it might be free to flow into the extra dimensions while the other forces would be stuck in our ordinary, four-dimensional spacetime. Thus, the strength of the other forces would be concentrated where we could measure it, while the strength of gravity would be diluted into the extra dimensions, where we cannot yet measure it [9].

A second motivation is a much more serious hierarchy problem: the cosmological constant. Naive estimates of the cosmological constant are off by factors as large as 10^{120}. It is possible that the vacuum energy could somehow dissipate into the extra dimensions [9].

2 The Scale Hierarchy Problem and The Randall-Sundrum Braneworlds

2.1 The Scale Hierarchy Problem

Traditionally there are at least two fundamental energy scales in particle physics, the electroweak scale, $M_{ew} = 246\ GeV$, and the Planck scale, $M_{Pl} = 1.22 \times 10^{19}\ GeV$. The electroweak scale is the scale at which electricity and the weak interaction are the same strength, while the Planck scale is the scale at which gravity and the gauge interactions are the same strength. The question of why these scales are so disparate is called the hierarchy problem.

In 1998, Arkani-Hamed, Dimopoulos, and Dvali (ADD) wrote a paper [10] suggesting that the weak scale may in fact be the only fundamental scale in nature. At the time of the ADD paper, while the electroweak interactions had been probed at distances on the order of $\frac{1}{M_{ew}}$, gravity had not yet been probed at any distance even remotely approaching $\frac{1}{M_{Pl}}$ [10]. Thus, the assumption of M_{Pl} as a fundamental energy scale required that gravity remain unmodified over more than thirty three orders of magnitude. But a 2001 paper by a collaboration from the University of Washington, Seattle [11] pushed the distance down by a factor of a thousand, leaving us in a situation in which gravity is *only* assumed to be unmodified over more than thirty orders of magnitude. Nonetheless, the ADD model remains relevant, if in a slightly modified form, and the original is important in understanding the variations on the Randall-Sundrum models that are the principal focus of our investigation.

The ADD model employs $N \geq 2$ extra dimensions to create a situation in which gravity and electromagnetism are of equal strength at the Electroweak scale, while gravity retains its ordinary $\frac{1}{M_{pl}}$ strength at larger distances. The ADD model is as follows [10].

Consider a world with n extra compact spatial dimensions. Give each of these dimensions a radius $\sim R$. Then, set the $4+n$-dimensional Planck scale on the order of the electroweak scale [10]

$$M_{Pl(4+n)} \sim M_{ew} \tag{1}$$

In this model gravity behaves very differently at distances smaller than the radius of the extra dimension, but behaves normally at distances larger than the radius of the extra dimensions.

Consider two test masses much closer to each other than the radius of the extra dimensions. Since the masses are well within the radius of the extra dimensions, the flux lines are free to propagate through the extra dimensions and the potential is given by Gauss' law in $4+n$ dimensions [10]

$$V(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2}} \frac{1}{r^{n+1}},\ (r \ll R). \tag{2}$$

But if the masses are much farther apart than the radius of the extra dimensions, the flux lines cannot travel through the extra dimensions to get from one mass to the other, so the potential is given by Gauss' law in 4 dimensions [10]

$$V(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2} R^n} \frac{1}{r}, \; (r \gg R) \tag{3}$$

If we set these two equations equal to each other for the $n = 0$ ($d = 4$) case, we see that the effective 4 dimensional M_{Pl} must be [10]

$$M_{Pl}^2 \sim M_{Pl(4+n)}^{2+n} R^n. \tag{4}$$

Then, we set the higher dimensional Planck scale on the order of the weak scale, $M_{Pl(4+n)} \sim M_{ew}$ as we did earlier, and solve for the radius of the extra dimensions so that the effective four-dimensional Planck scale matches observation [10]

$$R \sim 10^{\frac{30}{n} - 17} \mathrm{cm} \times \left(\frac{1 \mathrm{TeV}}{M_{ew}} \right)^{1 + \frac{2}{n}}. \tag{5}$$

For the case with only one extra dimension the radius would have to be on the order of $R \sim 10^{11}$ m in order for the Planck scale to be on the order of the weak scale, implying deviations from Newtonian gravity for distances on the order of the size of the solar system, so $n = 1$ is not possible. At the time of the ADD paper, for all $n \geq 2$ the modification of gravity would only take place at distances smaller than had previously been explored experimentally. ADD, then, put forth a model in which there were two compact spatial dimensions [10]. They also pointed out that experiments in preparation at the time of their paper would probe this scale, but those experiments [11] showed that gravity remains unmodified down to at least $218\mu m$, requiring more than two additional extra dimensions.

2.2 Randall-Sundrum 1

In 2006 Lisa Randall and Raman Sundrum introduced the first of their 'Randall-Sundrum' models. In this model the extra dimension is relatively small, but the mass hierarchy is large because the extra dimension is warped [7].

While the ADD paper offered one approach to the hierarchy problem, gravity ultimately proved to continue to exhibit $\frac{1}{M_{Pl}}$ behavior at distances smaller than allowed by the case with only two extra dimensions. Also, as Randall and Sundrum pointed out [7], in the ADD case the radius of the extra dimension would have to be much larger than $\frac{1}{M_{ew}}$, and since particle physics has been probed up to and beyond the weak scale the standard model particles and forces (that is, everything but gravity) must be confined to a four-dimensional subspace within the higher dimensional spacetime. Everything but gravity must be bound to a brane, otherwise we would have already experimentally detected the effects of the extra dimensions.

In the Randall-Sundrum 1 case, the spacetime is a slice of anti de Sitter space with a '3-brane' at each end.

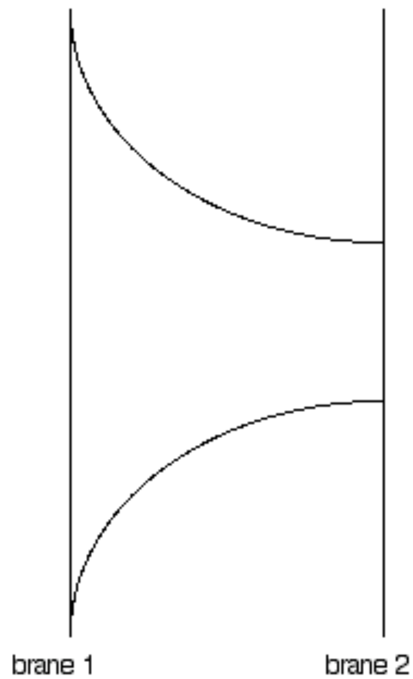

Figure 1: The Randall-Sundrum 1 Model. The lines show how the metric is warped along the extra dimension by the factor $e^{-A(z)}$.

2.3 Randall-Sundrum 2

While in their first collaboration Randall and Sundrum put forth a model that offered 'a large mass hierarchy from a small extra dimension,' [7] in their second collaboration they offer 'An Alternative to Compactification.' [8] This model is basically equivalent to the Randall-Sundrum 1 model, except that the second brane is taken to be infinitely far away from the brane with the gravitational bound state, effectively removing it from the spacetime. The extra dimension is then infinite in extent in both directions, but the spacetime appears, for the most part, to be effectively four-dimensional. Then scenario, then, overturns the previous assumptions that there must be precisely four non-compact dimensions, and that the higher dimensional Planck scale is directly proportional to the volume of the extra dimension. [8].

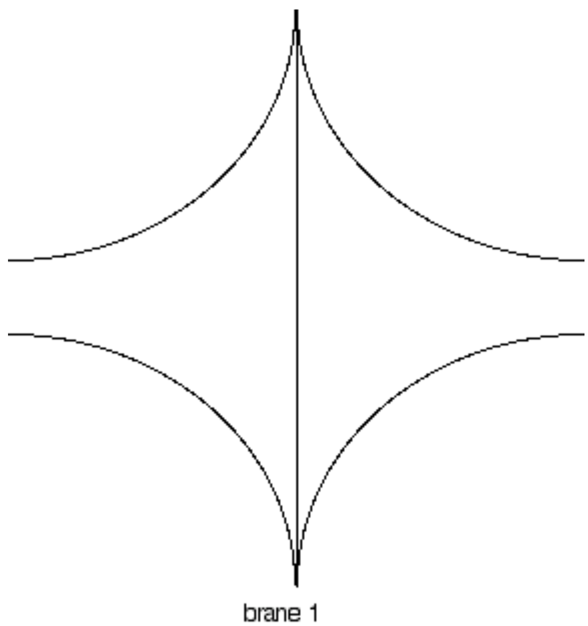

Figure 2: The Randall-Sundrum 2 Model. The lines show how the metric is warped along the extra dimension by the factor $e^{-A(z)}$.

3 Warp Duality

We consider whether it is possible that there are multiple five dimensional spacetimes that give rise to the same effective four-dimensional physics. In order to investigate this possibility, we begin with the most general class of five-dimensional conformally flat backgrounds which have a four-dimensional

Poincaré symmetry,

$$ds^2 = e^{-A(z)}\left(\eta_{ab}\, dx^a\, dx^b - dz^2\right) \ , \tag{6}$$

Where $a, b = 0, 1, 2, 3$ label the directions along the brane, and z labels the extra dimension. The fluctuations of this metric are [12]

$$-\tfrac{1}{2}\partial^\rho\partial_\rho\, \tilde{h}_{\mu\nu} + \left[\tfrac{9}{32}\partial^\rho A \partial_\rho A - \tfrac{3}{8}\partial^\rho\partial_\rho A\right]\tilde{h}_{\mu\nu} = 0 \ . \tag{7}$$

We then look for solutions of the form $\tilde{h}_{ab}(x,z) = \check{h}_{ab}(x)\psi(z)$ with $\Box_x \check{h}_{ab}(x) = m^2 \check{h}_{ab}(x)$, employing the fact that $\partial^\rho\partial_\rho = -\Box_x - \nabla_z^2$, where $\Box_x = -\eta^{ab}\partial_a\partial_b$ and $\nabla_z^2 = \partial_i^2$. to find: [12]

$$-\frac{d^2\psi(z)}{dz^2} + \left[\tfrac{9}{16}A'(z)^2 - \tfrac{3}{4}A''(z)\right]\psi(z) = m^2\psi(z) \tag{8}$$

This has the form of a Schrödinger equation for the "wavefunction" $\psi(z)$, "energy" m^2 and potential

$$V(z) = \tfrac{9}{16}(A'(z)^2 - \tfrac{3}{4}A''(z)) \tag{9}$$

.

The gravitational potential (as distinct from the Schrödinger potential) is given by [12]

$$U(r) \sim G_N \frac{M_1 M_2}{r} + \frac{1}{M_{Pl(5)}^{-3}} \frac{M_1 M_2 e^{-mr}}{r} \psi_m(0)^2 \tag{10}$$

When we examine the equation for the Schrödinger potential [9] we see that a second order differential equation on $A(z)$ gives the potential; thus there should be multiple solutions for $A(z)$ that give rise to the same potential. If we then take this potential and put it back into the 'Schrodinger' equation [8], we see that if the potentials are the same, the fluctuations are the same.

In order to begin with a potential that we know gives rise to a four-dimensional effective physics in agreement with experiment we first take the expression for $V(z)$ in the Randall-Sundrum 2 case [8].

$$V(z) = \frac{15k^2}{8(k|z|+1)^2} \tag{11}$$

And then substitute this expression in for $V(z)$ in [9] to obtain

$$\frac{15k^2}{8(k|z|+1)^2} = \frac{9}{16}\left(\frac{dA}{dz}\right)^2 - \frac{3}{4}\frac{d^2A}{dz^2} \tag{12}$$

Consequently, there is a range of possible $A(z)$'s, each corresponding to a different solution to [11], that all give rise to the same 'potential,' and since we showed that the four-dimensional gravitational modes are a function of this 'potential,' it is possible that there are multiple five dimensional warped spacetimes that give rise to the same effective four-dimensional physics.

In order to determine whether there are viable spacetimes that meet this requirement, I solved the above differential equation, first numerically, in order to obtain an idea of the behavior of the system, and then analytically.

The boundary conditions must be chosen to preserve the brane tension; thus, the total discontinuity in the derivative of the warp factor $(A(z))$ at zero must be constant, and is equal to $2k$. From this point forward, we choose units so that $k = 1$ for simplicity.

In order to test my code, I first evaluated the symmetrical, Randall-Sundrum 2 case, which generated figure 3.

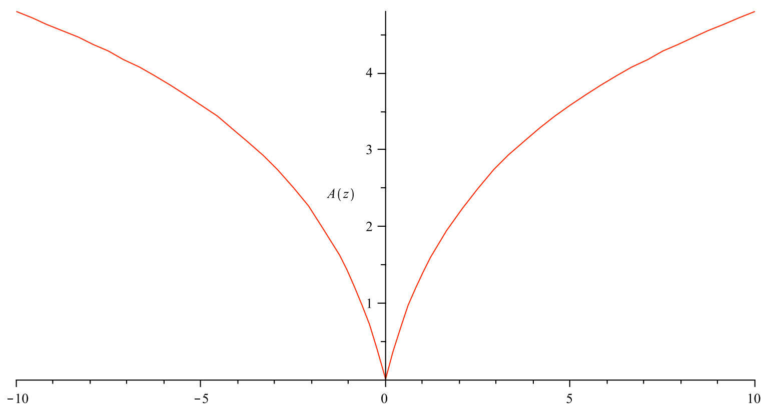

Figure 3: The Symmetrical Warp Factor

After then relaxing the requirement that the warp factor be symmetrical about the brane I obtained a family of plots with the appearance of figure 4.

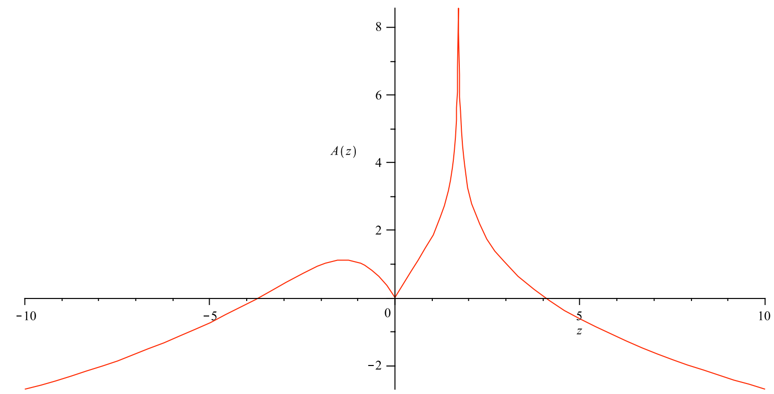

Figure 4: The Asymmetrical Warp Factor

Note the two distinct types of behavior on either side of the brane at z=0. It is also possible for this appearance to be flipped, depending on which side of the brane you decide to make steeper than the Randall-Sundrum 2 case. We discuss each of the behaviors in detail below.

3.1 $|A'(0)| > 2$

On the side where $A'(0) > 2$ the warp factor goes to infinity in finite distance. This means that the spacetime is cutting itself off in finite distance–that there is a boundary of the spacetime where the warp factor goes to infinity. Consequently, new boundary conditions will have to be imposed, and the spacetime will support different spacetime fluctuations.

In order to explore the behavior in this region further, we solve for the general solution and find

$$A(z) = \frac{16}{3}\ln(2) - \frac{2}{3}\ln\left(\frac{(-16 - 24z^3 + 12z^3 s_0 - 6z^4 + 3z^4 s_0 - 24z + 12z s_0 - 36z^2 + 18z^2 s_0)^2}{(z+1)^3}\right) \tag{13}$$

where s_0 is the value of $A'(0)$ from the right. The warp factor goes to infinity precisely where the argument of the natural log is equal to zero, namely

$$z_\infty = \frac{1}{3}\frac{27^{\frac{1}{4}}\left((10+3s_0)(-2+s_0)^3\right)^{\frac{1}{4}}}{-2+s_0} \tag{14}$$

Where z_∞ denotes the value of z for which a warp factor with a given s_0 goes to infinity. If we then plot this expression over the possible values of s_0 we obtain figure 5.

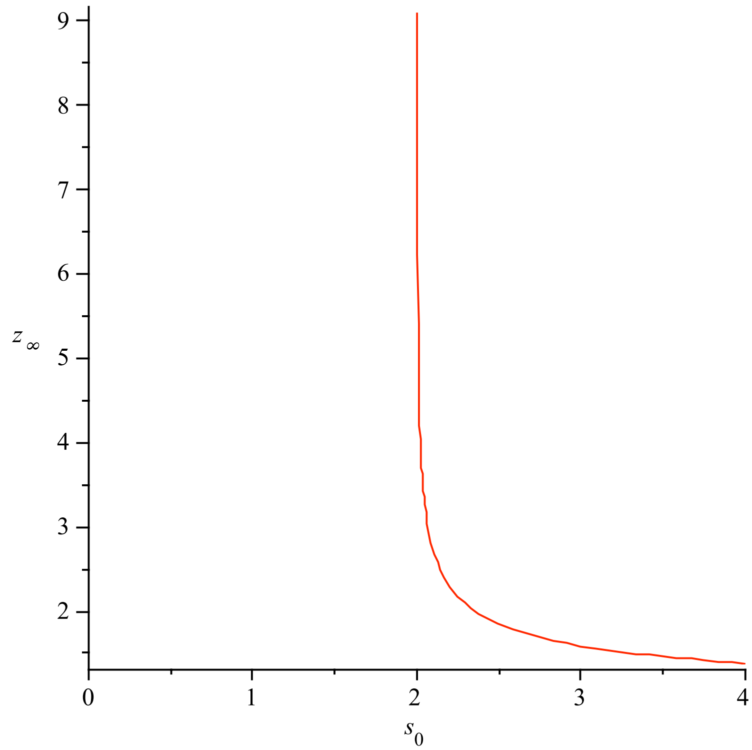

Figure 5: s_0 v. z_∞

Note that as we approach $s_0 = 2$, the value for the symmetrical, Randall-Sundrum case, the distance from the brane at which the warp factor blows up to infinity. Note also that for values of s_0 less than two this expression is undefined. The reason is that, in these cases, the warp factor exhibits the behavior seen in the $|A'(0)| < 2$ direction, which we discuss now.

3.2 $|A'(0)| < 2$

In the negative z direction of the above plot we see an example of the behavior of the warp factor when $|A'(0)| < 2$. This direction also appears problematic, primarily because it appears that there is a violation of the null energy condition, in which case the geometry of the spacetime is most likely unstable.

In order to determine whether there is a violation of the null energy condition, we have to determine whether $A''(y)$ is ever less than zero [13], where y is a coordinate related to z by [8]

$$z \equiv sgn(y)(\exp|y| - 1) \tag{15}$$

We then substitute this value for z into the general solution for this direction:

$$A(z) = \frac{2}{3}\ln(-\frac{256(z-1)^3}{(16 - 24z^3 - 12z^3 s_0 + 36z^2 + 18z^2 s_0 + 6z^4 + 3z^4 s_0 - 24z - 12zs_0^2)^2} \tag{16}$$

And obtain

$$A(y) = \frac{16}{3}\ln(2) + 2|y| + \frac{2}{3}\ln\left(\frac{1}{(10 + 6e^{4|y|} - 3s_0 + 3s_0 e^{4|y|})^2}\right) \tag{17}$$

In order to test this expression I first substituted in the symmetrical, Randall-Sundrum value for s_0 and then plotted it, obtaining figure 6, which agrees with the Randall-Sundrum 2 warp factor $A(y) = -2ky$. [7]

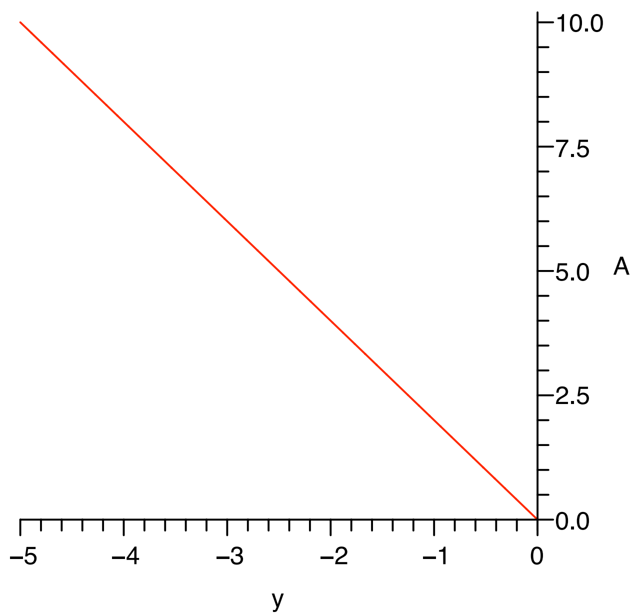

Figure 6: $A(y)$ (Randall-Sundrum 2 case). This is anti de Sitter space, with a negative constant curvature.

I then tested and plotted the asymmetrical case, obtaining figure 7, which clearly has a second derivative dipping below zero at some points, and is thus in violation of the null energy condition.

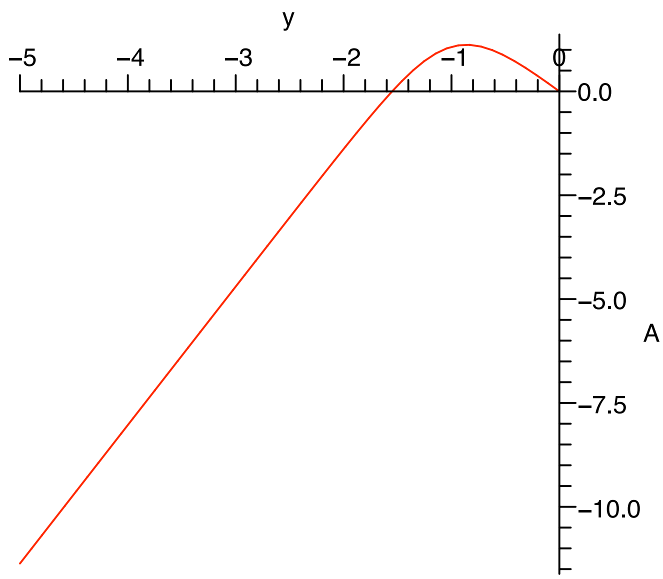

Figure 7: $A(y)$ (Asymetrical case). Note that $A''(y)$ dips below zero, violating the null energy condition.

4 Conclusions and Prospects for Future Research

Given the enormous problems discussed above with the candidate geometries, it appears that warp dualities of the kind sought are not viable candidates for our observed world. Nonetheless, it may be possible to construct related spacetimes that are viable candidates for the observed world. One possibility is to cut off the asymmetrical spacetimes with additional branes prior to the problematic behavior, but this approach necessarily changes the gravitational fluctuations because there will be the addition of numerous boundary conditions, and the warp factor is still highly unstable.

Another approach is to relax the requirement that the potential take the same value as that of the Randall-Sundrum case. In order for there to be four-dimensional gravity, there must be a normalizable, zero mass gravitational fluctuation. In order for this to happen, V(z) and A(z) are subject to the following restrictions [12]:

The zero-energy state is:

$$\psi_0(z) = \exp\Big[-\tfrac{3}{4}A(z)\Big] \ . \tag{18}$$

In order for this to be normalizable, $\exp[-\frac{3}{2}A(z)]$ must fall off faster than $\frac{1}{z}$ [12].

Moreover, if $V(z) > 0$ as $|z| \to \infty$, then $\hat{\psi}_0(z)$ is always normalizable. If $V(z) < 0$ as $|z| \to \infty$, then $\psi_0(z)$ is not normalizable and therefore cannot describe localized four-dimensional gravity. The $V(z) = 0$ as $|z| \to \infty$ case is perhaps the most interesting. The Randall-Sundrum 2 case falls into this category, as well as the other geometries I explored above [12].

These requirements are relatively open, so there are a great many possibilities to explore in the future.

References

[1] S. Deser, arXiv:physics/0402105.

[2] T. Kaluza,

[3] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921).

[4] O. Klein, Z. Phys. **37**, 895 (1926) [Surveys High Energ. Phys. **5**, 241 (1986)].

[5] M. J. Duff, arXiv:hep-th/9410046.

[6] S. Seahra http://etd.uwaterloo.ca/etd/ssseahra2003.pdf

[7] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-ph/9905221].

[8] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[9] C. Csaki, arXiv:hep-ph/0404096.

[10] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].

[11] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. **86**, 1418 (2001) [arXiv:hep-ph/0011014].

[12] C. Csaki, J. Erlich, T. J. Hollowood and Y. Shirman, Nucl. Phys. B **581** (2000) 309 [arXiv:hep-th/0001033].

[13] E. Witten, arXiv:hep-ph/0002297.