

# Minimal Length Uncertainty and the Quantum Mechanics of Non-Commutative Space-Time

A thesis submitted in partial fulfillment of the requirement for the degree of  
Bachelors of Science with Honors in Physics from The College of William  
and Mary

by

David E. V. Rose

Accepted for Honors

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Advisor: Prof. Chris Carone

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Prof. Jan Chaloupka

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Prof. John Delos

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Prof. Ilya Spitkovsky

Williamsburg, VA  
April 25, 2006

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## **Acknowledgements**

I would like to thank Professor Carone for his invaluable help on this project. It was a pleasure working with him. I would also like to thank Professor Chaloupka, Professor Delos, and Professor Spitkovsky for serving on my committee.

## Abstract

Modern physical theories such as string theory and theories of quantum gravity suggest the existence of a minimal observable physical length. A nonzero minimum spatial uncertainty can be incorporated into traditional non-relativistic quantum mechanics via a modification of the position operator. We explore the implications of this new quantum mechanics, which in essence has a spatial fuzziness when observed on the minimal length scale. In particular, we aim to bound the size of the minimal observable length via applications to atomic and elementary particle systems. Also, we study the effects of minimal length uncertainty on a free particle encountering a one-dimensional potential barrier.

# 1 Introduction

The need for a minimal observable length has been suggested in modern physical theories [9]. String theory suggests that matter is composed of vibrating strings which themselves have a fundamental size. Because of this, arbitrarily small distances cannot be probed via string-string scattering. In theories of quantum gravity, this minimal length regulates unwanted divergences. Consulting the traditional quantum mechanical uncertainty relation<sup>1</sup>

$$\Delta x \Delta p \geq \frac{1}{2} \quad (1.1)$$

we see after rearranging that

$$\Delta x \geq \frac{1}{2\Delta p}, \quad (1.2)$$

implying that position uncertainty can be made to vanish simply taking the limit  $\Delta p \rightarrow \infty$ . We thus see that the traditional formulation of quantum mechanics is an inadequate theory for dealing with this minimal observable length. It is easy to see that altering the uncertainty principle so that

$$\Delta x \Delta p \geq \frac{1}{2} (1 + \alpha^2 (\Delta p)^2), \quad (1.3)$$

where  $\alpha$  is a parameter with units of length, implies that  $\Delta x \geq \alpha$ . Note that it is a non-trivial assumption that a minimal observable physical length should be manifest as a minimal uncertainty in the position operator [8]. Nevertheless, consulting the generalized uncertainty principle [5] for two operators  $A, B$  acting on a Hilbert space, we see that

$$\Delta A \Delta B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|. \quad (1.4)$$

It is thus obvious from Eq. (1.4) that we must change our position and/or momentum operators in order to arrive at the uncertainty relation given in Eq. (1.3). Again from Eq. (1.4) we see that for our new operators  $x_{new}$  and  $p_{new}$  we have the equality

$$\frac{1}{2} (1 + \alpha^2 (\Delta p_{new})^2) = \left| \frac{1}{2i} \langle [x_{new}, p_{new}] \rangle \right|. \quad (1.5)$$

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<sup>1</sup>We follow the particle physics convention of setting  $\hbar = c = 1$ .

We should also note that we would like our new momentum and position operators to reduce to the traditional operators in the  $\alpha \rightarrow 0$  limit. A number of ways to satisfy Eq. (1.5) have been studied in the literature. We shall choose the simplest in which the momentum operator remains unchanged and the position operator remains Hermitian [4]. Our new operators are given by

$$x_{new} = x + \alpha^2 p x p \quad (1.6)$$

$$p_{new} = p \quad (1.7)$$

where  $x$  and  $p = -i \frac{d}{dx}$  are the traditional position and momentum operators. Other possibilities have been discussed in the literature such as

$$x_{new} = x + \alpha^2 p^2 x \quad (1.8)$$

$$p_{new} = p \quad (1.9)$$

given in [8] and the more general

$$x_{new} = x + \alpha^2 p^2 x + i\gamma p \quad (1.10)$$

$$p_{new} = p \quad (1.11)$$

which is given in [3]. Our choice has the natural generalization to three dimensions given by

$$x_{new}^i = x^i + \alpha^2 p^j x^i p^j \quad (1.12)$$

$$p_{new}^i = p^i \quad (1.13)$$

where the indices  $i$  and  $j$  range over all spatial coordinates and repeated indices are summed over according to the Einstein summation convention. We comment that these modifications to the position operators will also lead to a nontrivial commutation for the spatial coordinates:

$$[x^i, x^j] \neq 0 \text{ for } i \neq j. \quad (1.14)$$

Thus theories with minimal length uncertainty relations are noncommutative generalizations of traditional quantum mechanics. With our theory in place, we will now study quantum mechanical systems. In the past, general properties of the Hilbert space underlying these theories was studied in [8]. In more concrete applications, the hydrogen atom has been studied in both

[1] and [4]. In [1] and [8], a momentum space representation is utilized to analyze these problems. It is natural (but difficult) to work in momentum space as position wavefunctions lose some physical meaning in theories incorporating minimal length uncertainty. This is seen by noticing that we can approximate a position eigenstate as the limit of position space wavefunctions [8], however, a position eigenstate will contradict our minimal length uncertainty relation. We shall follow [4] in using perturbation theory to address many issues. We begin by applying our new physics to the diatomic molecule, in specific diatomic hydrogen, in an attempt to bound the size of the minimal length parameter  $\alpha$ . We also analyze the elementary particle system of quarkonium. In both these systems, we calculate the energy shifts due to our new physics and use them to bound the minimal length parameter. Finally, we will use numerical methods to study a one-dimensional free particle encountering a potential barrier.

## 2 The Diatomic Molecule

The traditional treatment of diatomic molecules is given in [2]. We begin with the Schrödinger equation for the relative motion of the nuclear masses  $M_1$  and  $M_2$  given by

$$\left[ \frac{-1}{2M} \nabla^2 + U(R) \right] \psi(R, \theta, \phi) = E\psi(R, \theta, \phi) \quad (2.1)$$

where  $R$  is the separation of the nuclei and  $M$  is the reduced mass given by

$$M = \frac{M_1 M_2}{M_1 + M_2}. \quad (2.2)$$

Utilizing the standard central force separation [5] we find that

$$\left[ \frac{-1}{2M} \frac{d^2}{dR^2} + U(R) + \frac{1}{2M} \frac{l(l+1)}{R^2} \right] u(R) = Eu(R) \quad (2.3)$$

where  $U(R)$  is the one-dimensional inter-nucleon potential,  $u(R) = Rr(R)$  for the radial wavefunction  $r(R)$ ,  $l$  is the azimuthal quantum number and  $R = x^i x^i$ , where the repeated indices are again summed over. The inter-nucleon potential can be approximated well for low electronic states by the Morse Potential, given by:

$$U(R) = U_0 \left[ e^{-2(R-R_0)/a} - 2e^{-(R-R_0)/a} \right]. \quad (2.4)$$

We can combine terms to write an effective one-dimensional potential as

$$U'(R) = U(R) + \frac{1}{2M} \frac{l(l+1)}{R^2} \quad (2.5)$$

which can be expanded about its minimum (at  $R = R_1$ ) to give:

$$U'(R) \simeq U'_o + \frac{1}{2}k(R - R_1)^2 + b(R - R_1)^3 + c(R - R_1)^4, \quad (2.6)$$

where

$$R_1 = R_0 + \frac{l(l+1)a^2}{2MR_0^3U_0} \quad (2.7)$$

$$U'_0 = -U_0 + \frac{l(l+1)}{2MR_0^2} - \frac{l^2(l+1)^2a^2}{4M^2R_0^6U_0} \quad (2.8)$$

$$k = \frac{2U_0}{a^2} - \frac{3l(l+1)}{MR_0^2a^2} \frac{a}{R_0} \left(1 - \frac{a}{R_0}\right) \quad (2.9)$$

$$b = -\frac{U_0}{a^3} \quad (2.10)$$

$$c = \frac{7U_0}{12a^4}. \quad (2.11)$$

We now observe that Eq. (2.6) appears like the potential to a simple harmonic oscillator with cubic and quartic perturbations. We can analyze the system utilizing perturbation theory. Recall that for a Schrödinger equation of the form

$$H\psi = E\psi \quad (2.12)$$

if we perturb the Hamiltonian so that  $H \rightarrow H + H'$ , then the first order shift in energy is given by

$$\Delta E_n = \langle \psi_n | H' | \psi_n \rangle \quad (2.13)$$

where  $E_n$  is the energy corresponding to the  $n$ th eigenstate  $\psi_n$ . The energy spectrum can be found in this way to give

$$E_n = U'_0 + \left(\frac{k}{M}\right)^{1/2} \left(n + \frac{1}{2}\right) - \frac{b^2}{Mk^2} \left[ \frac{15}{4} \left(n + \frac{1}{2}\right)^2 + \frac{7}{16} \right] \quad (2.14)$$

$$+ \frac{3c}{2Mk} \left[ \left(n + \frac{1}{2}\right)^2 + \frac{1}{4} \right]. \quad (2.15)$$

Our strategy now is to introduce our new position operators and retrieve the potential given in Eq. (2.6) plus additional terms which we will again treat with perturbation theory. We let

$$x_{new}^i = x^i + \alpha^2 p^j x^i p^j \quad (2.16)$$

for  $i, j = 1, 2, 3$ . We will discard terms of order  $O(\alpha^n)$  for  $n \geq 4$  throughout our consideration, as these terms are extremely small. We now have

$$R_{new} = \sqrt{x_{new}^i x_{new}^i} \quad (2.17)$$

$$= \sqrt{(x^i + \alpha^2 p^j x^i p^j)(x^i + \alpha^2 p^j x^i p^j)} \quad (2.18)$$

$$= \sqrt{x^i x^i + \alpha^2 (x^i p^j x^i p^j + p^j x^i p^j x^i)} + O(\alpha^4) \quad (2.19)$$

$$\simeq \sqrt{R^2 + \alpha^2 (x^i p^j x^i p^j + p^j x^i p^j x^i)} \quad (2.20)$$

$$= R \sqrt{1 + \frac{\alpha^2}{R^2} (x^i p^j x^i p^j + p^j x^i p^j x^i)} \quad (2.21)$$

$$\simeq R \left( 1 + \frac{\alpha^2}{2R^2} (x^i p^j x^i p^j + p^j x^i p^j x^i) \right) \quad (2.22)$$

$$= R + \frac{\alpha^2}{2R} (x^i p^j x^i p^j + p^j x^i p^j x^i). \quad (2.23)$$

Now let  $\beta = x^i p^j x^i p^j + p^j x^i p^j x^i$  to simplify notation. We will use the approximations

$$R_{new}^2 = R^2 + \alpha^2 \beta \quad (2.24)$$

$$R_{new} = R + \frac{\alpha^2}{2R} \beta. \quad (2.25)$$

We will now insert these new operators into our potential, aiming to retrieve our old potential, plus terms which can be viewed as additional perturbations. We begin with the quadratic term:

$$\frac{1}{2} k (R_{new} - R_1)^2 = \frac{1}{2} k (R_{new}^2 - 2R_1 R_{new} + R_1^2) \quad (2.26)$$

$$= \frac{1}{2} k \left( (R - R_1)^2 + \alpha^2 \left( 1 - \frac{R_1}{R} \right) \beta \right), \quad (2.27)$$

$$(2.28)$$

so our quadratic perturbation is

$$H'_2 = \frac{1}{2} k \alpha^2 \left( 1 - \frac{R_1}{R} \right) \beta. \quad (2.29)$$

Now we consider

$$R_{new}^3 = \left( R + \frac{\alpha^2}{2R} \beta \right)^3, \quad (2.30)$$

which can be expanded to give

$$R_{new}^3 = R^3 + \frac{\alpha^2 R}{2} \beta + \frac{\alpha^2}{2} \beta R + \frac{\alpha^2}{2R} \beta R^2 + O(\alpha^4) + O(\alpha^6). \quad (2.31)$$

Now our cubic term in the potential becomes

$$\begin{aligned} b(R_{new} - R_1)^3 &= b(R_{new}^3 - 3R_1 R_{new}^2 + 3R_1^2 R_{new} - R_1^3) \\ &= b \left[ (R - R_1)^3 + \alpha^2 \left( \frac{R}{2} - 3R_1 + \frac{3R_1^2}{2R} \right) \beta + \frac{\alpha^2}{2} \beta R + \frac{\alpha^2}{2R} \beta R^2 \right], \end{aligned} \quad (2.32)$$

giving a cubic perturbation:

$$H'_3 = b\alpha^2 \left( \frac{R}{2} - 3R_1 + \frac{3R_1^2}{2R} \right) \beta + \frac{b\alpha^2}{2} \beta R + \frac{b\alpha^2}{2R} \beta R^2. \quad (2.33)$$

We must now only consider the fourth order term. First, we see

$$R_{new}^4 = (R^2 + \alpha^2 \beta)^2 = R^4 + \alpha^2 R^2 \beta + \alpha^2 \beta R^2 + O(\alpha^4), \quad (2.34)$$

so the fourth order term in the potential is

$$\begin{aligned} c(R_{new} - R_1)^4 &= \\ &= c \left( (R - R_1)^4 + \alpha^2 R^2 \beta + \alpha^2 \beta R^2 - 2\alpha^2 R_1 R \beta \right. \\ &\quad \left. - 2\alpha^2 R_1 \beta R - \frac{2\alpha^2 R_1}{R} \beta R^2 + 6\alpha^2 R_1^2 \beta - \frac{4\alpha^2 R_1^3}{R} \beta \right). \end{aligned} \quad (2.35)$$

The quartic perturbation is thus given by

$$H'_4 = c\alpha^2 \left( R^2 - 2R_1 R + 6R_1^2 - \frac{2R_1^3}{R} \right) \beta - 2c\alpha^2 R_1 \beta R + c\alpha^2 \left( 1 - \frac{2R_1}{R} \right) \beta R^2. \quad (2.36)$$

We can combine (2.29), (2.33), and (2.36) to give the total perturbation:

$$H' = \tag{2.37}$$

$$\begin{aligned} & \alpha^2 \left( \frac{k}{2} - \frac{kR_1}{2R} + \frac{bR}{2} - 3bR_1 \right) \beta \\ & + \left( \frac{3bR_1^2}{2R} + cR^2 - 2cR_1R + 6cR_1^2 - \frac{2cR_1^3}{R} \right) \beta \\ & + \alpha^2 \left( \frac{b}{2} - 2cR_1 \right) \beta R + \alpha^2 \left( \frac{b}{2R} + c - \frac{2cR_1}{R} \right) \beta R^2. \end{aligned}$$

We now aim to simplify the quantity  $\beta$ . We have (where  $i$  is the imaginary unit when not an index)

$$\beta = x^i p^j x^i p^j + p^j x^i p^j x^i \tag{2.38}$$

$$= x^i (x^i p^j - [x^i, p^j]) p^j + p^j (p^j x^i + [x^i, p^j]) x^i \tag{2.39}$$

$$= x^i x^i p^j p^j - i\delta_{i,j} x^i p^j + p^j p^j x^i x^i + i\delta_{i,j} p^j x^i \tag{2.40}$$

$$= -R^2 \nabla^2 - \nabla^2 R^2 + i[p^i, x^i] \tag{2.41}$$

$$= -R^2 \nabla^2 - \nabla^2 R^2 + 3 \tag{2.42}$$

$$= -(R^2 \nabla^2 + \nabla^2 R^2 - 3), \tag{2.43}$$

following from the canonical commutation relation given by

$$[x^i, p^j] = i\delta_{i,j}. \tag{2.44}$$

We wish to compare to the pure vibrational spectrum ( $l = 0$ ), which is approximately given by:

$$E = - \left[ \sqrt{U_0} - \frac{1}{a\sqrt{2M}} \left( n + \frac{1}{2} \right) \right]^2. \tag{2.45}$$

Since  $Y_{00} = \left( \frac{1}{4\pi} \right)^{1/2}$ , the Laplacian in our perturbation will act only on the radial wavefunctions, we see that our energy shifts are given by

$$\Delta E_n = \langle u_n(R) | H' | u_n(R) \rangle, \tag{2.46}$$

where  $u_n(R)$  are the wavefunctions for an unperturbed simple harmonic oscillator. For  $t = \sqrt{M\omega}(R - R_1)$  and  $\omega = \sqrt{\frac{k}{M}}$ , the first few wavefunctions are given by:

$$u_0 = \left(\frac{M\omega}{\pi}\right)^{1/4} e^{-t^2/2} \quad (2.47)$$

$$u_1 = \left(\frac{M\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}} 2te^{-t^2/2} \quad (2.48)$$

$$u_2 = \left(\frac{M\omega}{\pi}\right)^{1/4} \frac{1}{2\sqrt{2}} (4t^2 - 2)e^{-t^2/2} \quad (2.49)$$

$$u_3 = \left(\frac{M\omega}{\pi}\right)^{1/4} \frac{1}{4\sqrt{3}} (8t^3 - 12t)e^{-t^2/2} \quad (2.50)$$

$$u_4 = \left(\frac{M\omega}{\pi}\right)^{1/4} \frac{1}{8\sqrt{6}} (16t^4 - 48t^2 + 12)e^{-t^2/2} \quad (2.51)$$

$$u_5 = \left(\frac{M\omega}{\pi}\right)^{1/4} \frac{1}{16\sqrt{15}} (32t^5 - 160t^3 + 120t)e^{-t^2/2} \quad (2.52)$$

$$(2.53)$$

and more generally by:

$$u_n(t) = \left(\frac{M\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(t) e^{-t^2/2}, \quad (2.54)$$

where  $H_n(t)$  is the  $n$ th Hermite polynomial. In order to compute the perturbations, we need to find numerical values for the parameters in the potential. With  $l = 0$ , the parameters of interest are  $U_0$ ,  $a$ , and  $R_0$ . Considering diatomic hydrogen, we consult [2] to find that  $R_0 = 375.012 \text{ MeV}^{-1}$ . We compare to the experimental data for the vibrational spectra to find the remaining two parameters. We extract the first few vibrational energies given in [7] in Table 1.

Using least squares methods, we fit the pure vibrational spectra given in Eq. (2.45) with this data to find the values  $U_0 = 4.73049 \text{ eV}$  and  $a = 258.759 \text{ MeV}^{-1}$ . Also,  $M = m_h/2$  where  $m_h = 938.272029 \text{ MeV}$ , the mass of a hydrogen nucleus. Using these values, we evaluate the inner products to find the values of our perturbations, summarized in Table 2. Considering the relatively large experimental error in the spectrum data, and the fact that

Quantum Number ( $n$ )	Vibrational Energy (in eV)	Error (in eV)
0	-4.46607	$\pm 6.05245 \times 10^{-6}$
1	-3.94957	$\pm 2.00196 \times 10^{-5}$
2	-3.46204	$\pm 4.26775 \times 10^{-5}$
3	-3.00326	$\pm 8.14753 \times 10^{-5}$
4	-2.57302	$\pm 1.43863 \times 10^{-4}$
5	-2.17109	$\pm 2.37287 \times 10^{-4}$

Table 1: Vibrational Energies for  $H_2$

$n$	$\Delta E_{n00}$
0	$-143.601 \text{ eV } (\alpha / \text{eV}^{-1})^2$
1	$-134.231 \text{ eV } (\alpha / \text{eV}^{-1})^2$
2	$-103.091 \text{ eV } (\alpha / \text{eV}^{-1})^2$
3	$-46.0552 \text{ eV } (\alpha / \text{eV}^{-1})^2$
4	$41.5616 \text{ eV } (\alpha / \text{eV}^{-1})^2$
5	$167.154 \text{ eV } (\alpha / \text{eV}^{-1})^2$

Table 2: Perturbations to the Hydrogen Spectrum

the perturbations are only on the eV scale, we do not expect a particularly strong bound on the minimal length parameter  $\alpha$ . Nevertheless, we can still approximate a bound. We can require the perturbations to the energy to be smaller than the error in the experimental data. Considering the ground state we thus have

$$143.601 \text{ eV} \times \left( \frac{\alpha}{\text{eV}^{-1}} \right)^2 \leq 2 \times 6.05245 \times 10^{-6} \text{ eV} = 1.21049 \times 10^{-5} \text{ eV}$$

giving a bound

$$\alpha \leq 290.337 \text{ MeV}^{-1}$$

or restoring  $\hbar$  and  $c$  to give standard units

$$\alpha \leq 5.73 \times 10^{-11} \text{ meters}$$

which is a weaker bound than given in the literature [4]. A slightly tighter bound can be found using data from [10]. In this paper the ground state

energy for molecular hydrogen is measured accurately to  $\pm 1 \times 10^{-8}$  eV which gives

$$143.601 \text{ eV} \times \left( \frac{\alpha}{\text{eV}^{-1}} \right)^2 \leq 2 \times 10^{-8} \text{ eV}$$

giving a bound for  $\alpha$  of

$$\alpha \leq 11.801 \text{ MeV}^{-1}$$

or

$$\alpha \leq 2.33 \times 10^{-12} \text{ meters.}$$

Although our quantitative analysis does not enable us to bound  $\alpha$  tightly, we still can comment qualitatively. In Figure 1 we have plotted  $E(\alpha)$  as a function of  $\alpha$ .

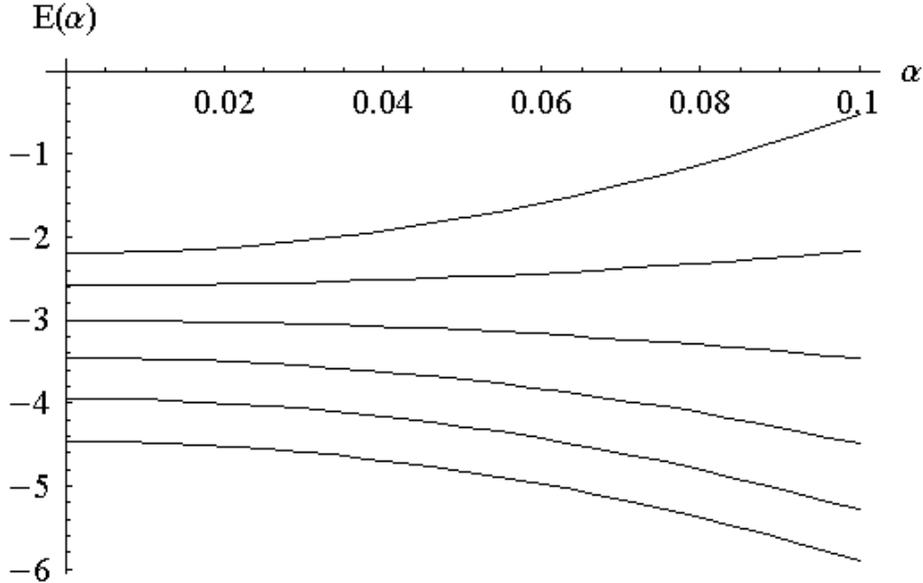


Figure 1:  $E(\alpha)$  versus  $\alpha$  in eV

Note that for the first four energy levels, the energy spectrum shifts negatively, but then a sign change occurs and the next few energy levels shift positively. This behavior is expected, as in [4] it was shown that the perturbations to the energy levels for a simple harmonic oscillator grow as a function of the principal quantum number. Also, note that Eq. (2.45) implies that the energy spacing between quantum levels decreases as the quantum level

increases. With sufficiently precise data, it is our hope that we can exploit these counteracting effects to bound the minimal length parameter tightly. Unfortunately, the energy data we have presented is the most precise found in the literature. To estimate the precision needed in the spectrum of diatomic hydrogen to give an interesting bound, we consider a plot of  $\frac{|E(\alpha)-E(0)|}{E(0)}$  versus  $\frac{1}{\alpha}$  given in Figure 2.

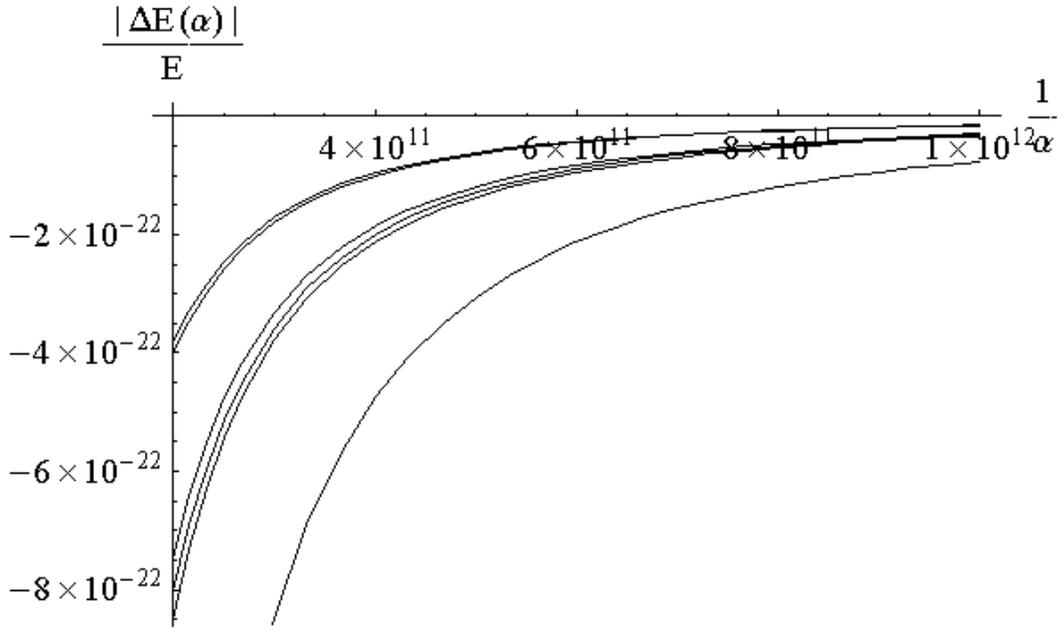


Figure 2:  $\frac{|E(\alpha)-E(0)|}{E(0)}$  versus  $\frac{1}{\alpha}$  in eV

We see that in order to bound  $\alpha$  on the TeV scale, we would need precision in the diatomic hydrogen spectra data better than  $10^{-22}$  eV. We are interested in energy scales on the order of TeV not only because such a bound would be tighter than those found in the literature, but also because this scale corresponds to the energies of the current highest energy particle colliders. It is our hope that these results will motivate future experiments to more accurately measure the vibrational spectrum of diatomic hydrogen.

### 3 Quarkonium

The elementary particle system of a quark-antiquark pair known as quarkonium has been modeled using non-relativistic quantum mechanics. The Hamiltonian for this system is given in [6] and appears hydrogen-like with a number of perturbative terms, as well as a linearly confining term. Dropping the perturbative terms, we find the potential to be given by

$$V(r) = \frac{-4a_s}{3r} + Ar$$

with the parameters  $a_s$  (the strong coupling constant) and  $A$  as given in [6]. Recall that the Coulomb potential is given by

$$V(r) = \frac{-e^2}{r}$$

so we can utilize the energy shifts found in [4] for the hydrogen atom simply by making the parameter substitution  $e^2 \rightarrow \frac{4a_s}{3}$ , and then viewing the linear term as a perturbation. The energy shifts due to minimal length uncertainty for the hydrogen spectrum were found to be

$$\begin{aligned} \Delta E_{100} &= \frac{5e^2}{a_0^2} \alpha^2 \\ \Delta E_{200} &= \left( 19 + \frac{\sqrt{12544 + 18\pi^2}}{8} \right) \frac{e^2}{48a_0^3} \alpha^2 \\ \Delta E_{21\pm 1} &= \left( 19 - \frac{\sqrt{12544 + 18\pi^2}}{8} \right) \frac{e^2}{48a_0^3} \alpha^2 \\ \Delta E_{210} &= \frac{5e^2}{48a_0^3} \alpha^2 \end{aligned}$$

with  $a_0 = \frac{1}{\mu e^2}$  where  $\mu = \frac{m_e m_p}{m_e + m_p}$  is the reduced mass of the system ( $m_e$  and  $m_p$  are the electron and proton mass respectively). It follows that taking  $a_0 \rightarrow \frac{1}{\mu(\frac{4a_s}{3})}$  and  $\mu = \frac{m_q}{2}$  for quark mass  $m_q$  gives the shifts for quarkonium

due to the hydrogen-like term:

$$\begin{aligned}\Delta E_{100} &= \frac{160m_q^3 a_s^4}{81} \alpha^2 \\ \Delta E_{200} &= \left(19 + \frac{\sqrt{12544 + 18\pi^2}}{8}\right) \frac{2m_q^3 a_s^4}{243} \alpha^2 \\ \Delta E_{21\pm 1} &= \left(19 - \frac{\sqrt{12544 + 18\pi^2}}{8}\right) \frac{2m_q^3 a_s^4}{243} \alpha^2 \\ \Delta E_{210} &= \frac{10m_q^3 a_s^4}{243} \alpha^2.\end{aligned}$$

For the shifts due to the linear term, we use perturbation theory on hydrogen-like wavefunctions. These functions are given by

$$\psi_{nlm}(r, \theta, \phi) = \sqrt{\left(\frac{2}{nb}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-r/nb} \left(\frac{2r}{nb}\right)^l [L_{n-l-1}^{2l+1}(2r/nb)] Y_l^m(\theta, \phi)$$

where  $b = \frac{3}{4\mu a_s}$ ,  $L_j^i$  are associated Laguerre polynomials, and  $Y_j^i$  are spherical harmonics [5]. We now recall from our previous discussion that

$$r_{new} = r + \frac{\alpha^2}{2r} \beta$$

so our perturbation in the energy due to minimal length uncertainty resulting from the linear term is given by

$$\Delta \tilde{E}_{nlm} = \langle \psi_{nlm} | A \frac{\alpha^2}{2r} \beta | \psi_{nlm} \rangle.$$

We find that

$$\begin{aligned}\Delta \tilde{E}_{100} &= 0 \\ \Delta \tilde{E}_{200} &= \frac{m_q A a_s}{4} \alpha^2 \\ \Delta \tilde{E}_{21\pm 1} = \Delta \tilde{E}_{210} &= \frac{5m_q A a_s}{12} \alpha^2.\end{aligned}$$

We now consider the systems of charmonium (a charm quark-antiquark pair) and bottomonium (a bottom quark-antiquark pair). For charmonium, we

	Charmonium Shift	Bottomonium Shift
$\Delta E_{100}$	0.0816 GeV $(\alpha/ \text{GeV}^{-1})^2$	1.50 GeV $(\alpha/ \text{GeV}^{-1})^2$
$\Delta E_{200}$	0.0319 GeV $(\alpha/ \text{GeV}^{-1})^2$	0.2656 GeV $(\alpha/ \text{GeV}^{-1})^2$
$\Delta E_{21\pm 1}$	0.0363 GeV $(\alpha/ \text{GeV}^{-1})^2$	0.1318 GeV $(\alpha/ \text{GeV}^{-1})^2$
$\Delta E_{210}$	0.0364 GeV $(\alpha/ \text{GeV}^{-1})^2$	0.1324 GeV $(\alpha/ \text{GeV}^{-1})^2$

Table 3: Energy Shifts for Bottomonium and Charmonium

have  $m_q = 1.2$  GeV,  $a_s = 0.392$ , and  $A = 0.177$  GeV<sup>2</sup> and for bottomonium we have  $m_q = 4.78$  GeV,  $a_s = 0.288$ , and  $A = 0.177$  GeV<sup>2</sup>. Combining our two energy shifts, we summarize the results in Table 3. Although precise experimental data is not available, we can still place an approximate bound on the size of the minimal length parameter  $\alpha$ . The ground state energy for bottomonium is known to be approximately  $E_{100} = 9$  GeV. Assuming this energy can be measured with ten percent accuracy, we have

$$1.5 \text{ GeV} \times \left( \frac{\alpha}{\text{GeV}^{-1}} \right)^2 \leq 0.9 \text{ GeV}$$

giving a bound of

$$\alpha \leq 0.775 \text{ GeV}^{-1}.$$

or simply

$$\alpha \leq 1.53 \times 10^{-16} \text{ meters.}$$

While this estimate is rough, it is much stronger than the bound placed using diatomic hydrogen due to the high energy scale of the quarkonium system. It is our hope that this preliminary consideration will motivate future research into a more rigorous consideration of the application of minimal length uncertainty to the quarkonium system, possibly in the framework of relativistic quantum mechanics.

## 4 One Dimensional Barrier

In a typical course in quantum mechanics, the system of a free particle encountering a one-dimensional barrier is studied. In such a problem, we have a potential given by

$$V(x) = \begin{cases} V_0 & \text{if } 0 \leq x \leq a \\ 0 & \text{otherwise.} \end{cases}$$

It is not obvious how to introduce minimal length uncertainty into this problem as the Hamiltonian does not explicitly depend on position. Instead, we choose to analyze the one-dimensional system given by

$$V(x) = \begin{cases} Ax + V_0 & \text{if } 0 \leq x \leq a \\ 0 & \text{otherwise.} \end{cases}$$

Taking the limit  $A \rightarrow 0$  will reduce to our original problem, and the system will be a good approximation to the constant barrier provided the slope  $A$  is not too large. The Schrödinger equation for this system is given by

$$\frac{-1}{2m} \frac{d^2\psi}{dx^2} = E\psi$$

in the regions where the potential is zero. The solutions are well known to be plane waves given by

$$\psi(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

where  $k = \sqrt{2mE}$ . In the intermediate region, we have the equation

$$\frac{-1}{2m} \frac{d^2\psi}{dx^2} + Ax\psi + V_0\psi = E\psi. \quad (4.1)$$

We now introduce minimal length uncertainty, taking  $x_{new} = x + \alpha^2 p_x p_x$ . This leads to the following Schrödinger equation in the intermediate region:

$$\frac{-1}{2m} \frac{d^2\psi}{dx^2} + Ax\psi - \alpha^2 \frac{d}{dx} \left( x \frac{d\psi}{dx} \right) + V_0\psi = E\psi.$$

Expanding, we see this equation takes the form

$$\frac{-1}{2m} \frac{d^2\psi}{dx^2} + Ax\psi - \alpha^2 \frac{d\psi}{dx} - \alpha^2 x \frac{d^2\psi}{dx^2} + (V_0 - E)\psi = 0. \quad (4.2)$$

Traditionally, our method for analyzing this system would be to solve equation (4.1) and then match the boundary conditions to the solutions where the potential is zero. If we assume an incident particle traveling from the left, these solutions take the form

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & \text{for } x < 0 \\ Te^{ikx} & \text{for } x > a \end{cases}$$

where we have fixed  $c_1$  by normalization, and assumed that there is no incident wave from the right [5]. The quantities of interest are then given by  $|R|^2$ , the reflection coefficient, and  $|T|^2$ , the transmission coefficient, representing the probabilities that the wave is reflected and transmitted respectively. Unfortunately, there is no simple solution to equation (4.2), so we must proceed numerically. To do so, we alter our strategy slightly. Traditionally, we matched at the boundary by setting

$$\begin{aligned} \psi_{left}(0) &= \psi_{middle}(0) \\ \psi'_{left}(0) &= \psi'_{middle}(0) \\ \psi_{middle}(a) &= \psi_{right}(a) \\ \psi'_{middle}(a) &= \psi'_{right}(a). \end{aligned}$$

Since we do not know the solution in the intermediate region, we must numerically search for the best solution by testing a variety of values for  $R$ . We solve the differential equation (4.2) with initial conditions  $\psi(0) = 1 + R$  and  $\psi'(0) = ik(1 - R)$ , the analog of matching boundary conditions at  $x = 0$ . To match boundary conditions at  $x = a$ , we search for a solution such that  $\psi'(a) = ik\psi(a)$ . This calculation is carried out using Mathematica, with sample code below:

```
m = 1/2;
Energy = 2;
V = 1;
A = 1/100
a = 10
theta = 0;
k = sqrt(2*m*Energy);
alpha = 1;
```

```

B = 0;
Do[
  R = 0;
  Do[
    Clear[F];
    Clear[B];
    Clear[solution];
    solution =
    NDSolve[{-1/(2m)*y''[x] + A*x*y[x] - alpha^2*y'[x]
    - alpha^2*x*y''[x] + (V - Energy)*y[x] == 0,
    y[0] == 1 + Re^(i*theta), y'[0] == i*k*(1 - Re^(i*theta))},
    y, {x, 0, a}];
    F = y[a]/.solution;
    B = Min[Abs[(i*k*y[a]/.solution)-(y'[a]/.solution)],
    0.01];
    If[B!=0.01, Print[B, Abs[R], Re^(i*theta), Abs[F],F]];
    R=R + 1/200, {i,200}];
  theta = theta + pi/256, {i,512}]

```

Note that we input values for  $R$  such that  $|R| \leq 1$ , as it would be unreasonable for the probability of the wave being reflected to be greater than 1. In essence, we stratify the unit disk and search for the value for  $R$  such that  $\psi'(a)$  best matches  $ik\psi(a)$ . The code outputs all values for the difference of these two values which are sufficiently small (about 1 – 2 percent of  $\psi'(a)$ ). We then choose the smallest and take the corresponding values for  $|R|$  and  $|F|$  to analyze the system.

We now pose the question, what happens when the length of the barrier  $a$  shrinks smaller than the minimal length parameter  $\alpha$ ? To analyze this situation, we simply run the code above for a variety of values of  $a$ , keeping the other parameters constant. We let  $m = 1/2$ , Energy = 2,  $V_0 = 1$ ,  $\alpha = 1$ , so all our energy scales are comparable, and take  $A = \frac{1}{100}$ , so our barrier is relatively flat. The values for  $|R|$  and  $|T|$  are given in table 4

We plot this data in Figure 3 to easily observe the overall behavior of the transmission and reflection coefficients as the width of the barrier varies.

Notice that as the barrier shrinks to a size smaller than that of the minimal length,  $|T|$  (and thus  $|T|^2$ ) approaches 1, while  $|R|$  (and  $|R|^2$ ) approaches zero. It seems that when the barrier is smaller than the fundamental length, the incident wave ‘sees’ less of the barrier, and thus travels right over the

$a$	$ R $	$ T $
10	0.775	0.197
9	0.740	0.212
8	0.695	0.241
7	0.620	0.277
6	0.505	0.326
5	0.415	0.372
4	0.460	0.396
3	0.545	0.419
2	0.560	0.480
1	0.435	0.636
0.5	0.265	0.788
0.25	0.145	0.884
0.1	0.060	0.952
0.01	0.005	0.996

Table 4:  $|R|$  and  $|T|$  for various values of  $a$

barrier with nearly perfect transmission.

Taking a more careful look at the data in Table 4 we see a peculiar result. Notice that the transmission coefficients are  $|R|^2$  and  $|T|^2$ . In the traditional case, it was always the case that  $|R|^2 + |T|^2 = 1$ . This is the conservation of probability current for the system, which can be derived from the Schrödinger equation. However, our numerical solution appears to show that we are altering the Schrödinger equation in such a way as to alter the conservation of probability current. To check that this is not simply a flaw in the numerical methods, we took the limit  $\alpha \rightarrow 0$  in the code and ran similar simulations for this potential as well as many other shapes. In each of these cases, probability current was verified to be conserved. This peculiarity suggests that either probability current is not conserved in this theory or that the statistical interpretation is not entirely valid. It is our hope that future work in this direction will shed light on this phenomenon.

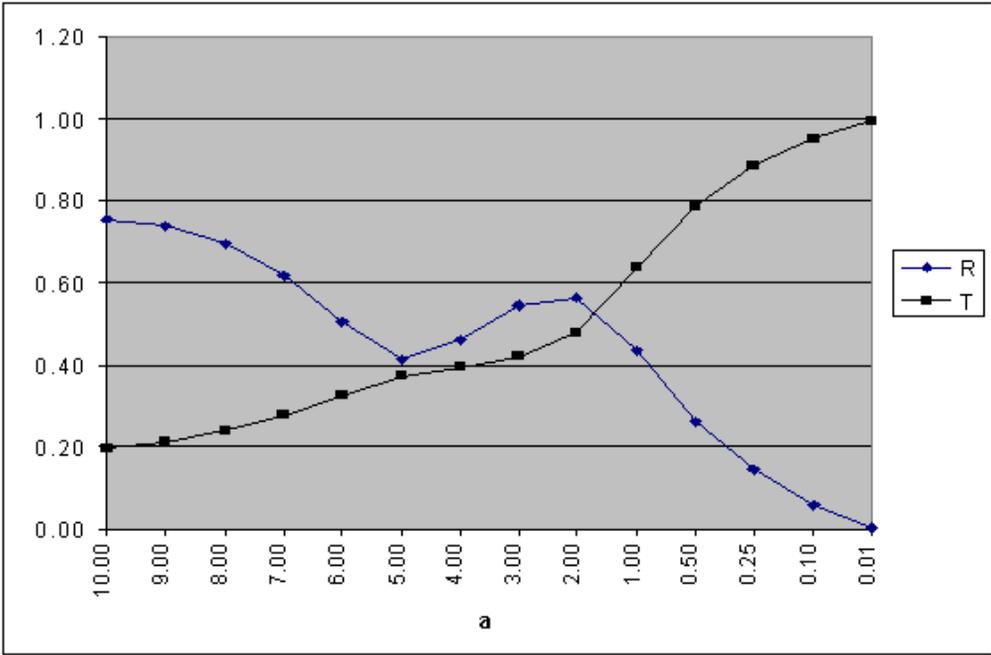


Figure 3:  $|R|$  and  $|T|$  as a function of  $a$

## 5 Conclusion

We were able to successfully explore the quantum mechanical structures of a space in which there is a nontrivial lower bound on position uncertainty. Through applications to molecular and elementary particle systems, we were able to not only bound the minimal length parameter  $\alpha$ , but also to motivate the experimental investigation of precise data on diatomic and quarkonium spectra. Finally, we were able to explore the underlying effects of minimal length uncertainty of the traditional structure of quantum mechanics by exploring the one-dimensional barrier. Hopefully, this investigation will prove helpful in future investigations of scattering in minimal length theories.

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