Exploring Gravity in Extra Dimensions

A thesis submitted in partial fulfillment of the requirement
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Physics from the College of William and Mary in Virginia

by

Kelly Sullivan

Accepted for ______________________________
(Honors)

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Advisor: Professor Sher

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Professor Hoatson

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Professor Leemis

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Professor Carone

Williamsburg, Virginia
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Abstract

One of the newest topics in theoretical particle physics is the idea of using compact extra dimensions to solve the hierarchy problem. It is proposed that by using compact submillimeter dimensions that only gravitons can 'sense', the functional form of the gravitational force will change in a way that will give it comparable strength at high interaction energies to the other three fundamental forces. While the functional dependence of gravity has already been calculated for distances much greater than and much less than the size of the extra dimensions, the aim of my thesis will be to find the functional dependence at distances approximately equal to the size of the dimensions, and from there to investigate other possible configurations, e.g. allowing the dimensions to be different lengths or meet at non-orthogonal angles. Future work will depend on the results of the first two parts.

Contents

1 Introduction

1.1 The Hierarchy Problem

Theories involving extra dimensions began in the 1920's, with the work of Theodor Kaluza and Oskar Klein. Together they found that a unified theory of gravity and electromagnetism was possible only under the condition that a fifth compact (or "curled up") dimension exists. From there, other motivations, including a quantization of gravitational interactions, led to increasingly more theories involving extra dimensions. String Theory, a candidate theory for quantum gravity, requires that we live in a world of ten space-time dimensions, six of which are curled up into a small compact space [2]. However, the size of these extra dimensions is extremely small. Quantum gravitational effects become evident at the energy scale at which the Compton wavelength of a particle becomes smaller than the particle's Schwarzchild radius. This energy scale, which is known as the Planck scale, is approximately 10^{19} GeV, corresponding to a length of 10^{-33} cm. This is a significantly higher energy than any range of particle accelerators, making experimental detection of extra dimensions quite improbable, if not impossible.

The reason that the Planck scale is so high relates to the fact that gravity is so much weaker than the other three fundamental forces at macroscopic scales. For example, two electrons would have to be 10^{22} times more massive for the gravitational force between them to match the Coulomb force between them [7]. Gravity's comparable weakness means that huge energies must be reached before the strength of gravity even approaches the strength of the strong, weak and electromagnetic forces. This disparity between the Planck scale and the weak scale (the scale of the masses of the other force carriers, $\sim 10^3$ GeV) is what is known as the hierarchy problem. The present version of the Standard Model is incapable of explaining the hierarchy problem, but physicists have come up with several solutions, one of which involves the use of large compact dimensions.

1.2 Brane Worlds

According to string theory, the world as we know it is merely a 3-dimensional brane (short for membrane) in a higher dimensional reality, the way that a plane is a 2-dimensional membrane in a 3-dimensional universe [2]. One way to visualize this idea is to think of a cylinder. With one extra spatial dimension, our 3-dimensional world is one line on that cylinder, while the extra compact dimension is the circle that connects that line back to itself (Figure 1).

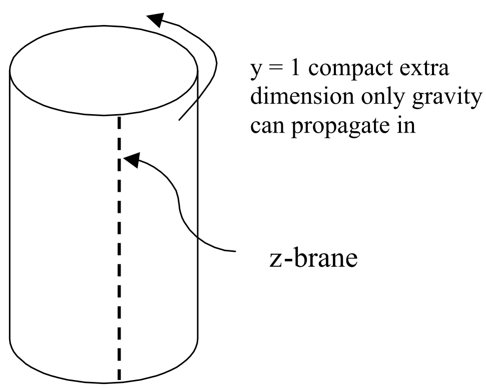

Figure 1: With one compact extra dimension, our three dimensional world acts like one line on a 4-dimensional cylinder

String theory contains dynamical objects that extend into p dimensions, but Dirichlet boundary conditions require that the ends of open strings that represent these objects must be located somewhere [2]. The 3-brane that we live on is therefore home to the ends of open strings that represent the force particles of the strong, weak, and electromagnetic forces. Gravity, however, which is represented by a closed loop, lies in the bulk (i.e. it is not stuck to the brane), and so gravity extends in all dimensions.

Models in which gravity is not constrained to three dimensions have the capability of decreasing the Planck scale because the force of gravity becomes stronger as the number of dimensions increases [1]. Using Gauss' Law, we can easily find that with one extra compact dimension, the gravitational force changes from $\frac{1}{r^2}$ form to $\frac{1}{r^3}$ form. Therefore, at small distances, the strength of gravity increases more quickly, making it comparably strong to the other forces at an energy closer to that of the weak scale. The fact that they are *compact* means that at macroscopic distances we can no longer see the extra dimensions and gravity takes its familiar form.

Though we know that string theory requires six extra spatial dimensions, we still don't know how many of them will be large. If the Planck scale were to be reduced to 1 TeV, it has been found that the size of the extra dimensions would be $R \sim 10^{32/n-17}$ cm, where n is the number of additional dimensions. For $n = 1$, this places the radius of the extra dimension at 10^{15} cm, which clearly is not the case in real life. For $n = 2$, the radii of the extra dimensions are in the range of hundreds of microns, and for $n = 3$, the radii would be in the nanometer range. Three large extra dimensions would also provide a nice symmetry, with three macroscopic, three intermediate, and three small dimensions. For more than three extra dimensions, we return to the problem that the dimensions are too small to experimentally detect.

The gravitational potential in n dimensions has been calculated for distances small enough to be sufficiently encased in the extra dimensions, and gravity at macroscopic distances has been known for hundreds of years. What has not been calculated is the potential at distances of the same order of magnitude as the extra dimensions, precisely what experimenters would see first if these dimensions exist. My thesis explores gravity at these distances, and explores what might happen to the potential for different configurations for the extra dimensions. For example, with two extra dimensions, the radii do not have to be the same, nor do they have to meet at a

90^o angle. The don't they even have to exist on a flat space; they could just as conceivably exist on the surface of a sphere. I first will go through calculations of simpler topologies using Gauss' Law, and then will move on to the calculation of the potential on spherical extra dimensions.

2 Calculation Methods using Gauss' Law

To calculate the functional form of gravity in extra dimesnions, we can use the n-dimensional form of Gauss' Law. For example, for $n = 3$, the gravitational form of Gauss' Law is:

$$\int F \cdot da = 4\pi G_N^{(3)} M_{\text{enc}}, \tag{1}$$

where $G_N^{(3)}$ is the 3-dimensional Newton's constant and M_{enc} is the total mass enclosed in the Gaussian surface. We find that the LHS is $4\pi r^2$, and the RHS equals a constant times G times the mass enclosed. The constant of 4π on the RHS is the surface area of a 3-dimensional unit sphere, which cancels the 4π on the other side. In extra dimensions, Gauss' law remains basically the same, except with n-dimensional spherical surfaces instead of 3-dimensional surfaces. It has been calculated [4] that the surface area of an n-dimensional unit sphere is:

$$S_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \tag{2}$$

and so Gauss law becomes:

$$\int F \cdot da = S_n G_N^{(n)} M_{\text{enc}}. \tag{3}$$

For example, in four dimensions, the surface area of a sphere ($\int da$ on the LHS) is

$2\pi^2 r^3$, and so gravity has a $\frac{1}{r^3}$ dependence. For five and six dimensions, the forces are proportional to $\frac{1}{r^4}$ and $\frac{1}{r^5}$, respectively.

The above equations only work when $R << L$, that is, when the distance we are working with is sufficiently encased in the compact dimensions. To return to our standard $\frac{1}{r^2}$ law, we zoom out so that $R >> L$. Note that the nature of the compact dimensions is that once you traverse the length L, you wind up back in the same place you started (Figure 2).

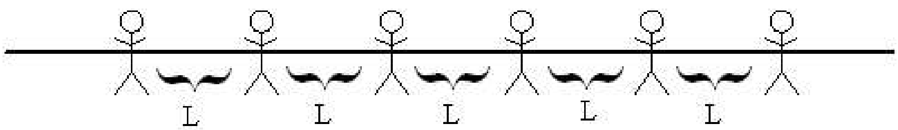

Figure 2: Due to compactification, a single object in one extra dimension will instead look like an infinite line of objects.

Thus, from a distance, one would see an infinite line of points with one compact dimension, a plane of points in two compact dimensions, and so on. So with one extra dimension, when $R >> L$ we can assume that we are far enough away that the single mass looks like a continuous n-dimensional "line" of uniform mass density, and we can use a cylindrical Gaussian surface to solve for the force. Let an n-dimensional cylinder of side length l enclose a line of mass $M(\frac{l^n}{L^n})$ [1]. The endcaps of our cylinder are 3-dimensional spheres with radius r, and we use the n-dimensional Gauss' law, where the area enclosed is $l^n(4\pi r^2)$. Under these conditions, Gauss' law becomes:

$$F(4\pi r^2 l^n) = S_n G_N^{(n)} M \left(\frac{l^n}{L^n}\right). \tag{4}$$

Lumping constants on both sides together into $G_N^{(3)}$, we can solve this equation for F:

$$F = \frac{G_N^{(3)} M}{r^2}, \tag{5}$$

and so we get our original $\frac{1}{r^2}$ law back, with the 3-dimensional Newton's constant in terms of the higher-dimensional $G_N^{(n)}$ and S_n:

$$G_N^{(3)} = \frac{S_{3+n}}{4\pi} \frac{G_N^{(3+n)}}{V^n}, \tag{6}$$

where now n is the number of *extra* dimensions, and $V_n = L^n$ is the volume of the dimensions. As an example, take the five dimensional case. The surface area of a unit 5-dimensional sphere is $\frac{2\pi^{5/2}}{\Gamma(5/2)} = \frac{8\pi^2}{3}$, and so Eq. (6) can be solved for $G_N^{(5)}$:

$$G_N^{(5)} = \frac{4\pi V_2 G_N^{(3)}}{S_5} = \frac{3GL^2}{2\pi}. \tag{7}$$

By the same process, we find that

$$G_N^{(4)} = \frac{2G_N^{(3)} L}{\pi}, \qquad G_N^{(6)} = \frac{4G_N^{(3)} L^3}{\pi^2}. \tag{8}$$

The cases where $R >> L$ and $R << L$ have been fully analyzed, but little attention has been paid to the case where $R \approx L$. This is the subject of my project. When $R \approx L$, we can't assume that we are far enough away that the line of points looks like a line of continuous charge density. Instead the potentials of each mirror image must be summed.

3 One Extra Dimension

To do the calculation associated with these infinite sums, I employed Mathematica, but I first calculated on paper what the sum for the gravitational potential in extra compact dimension should be. Using the 4-dimensional Gauss' Law, I first calculated the gravitational potential from the force, so that vector components wouldn't be an issue:

$$V(r) = -\int \frac{G_N^{(4)} Mm}{r^3} dr = \frac{G_N^{(4)} Mm}{2r^3} = \frac{G_N^{(3)} LMm}{\pi r^2} \tag{9}$$

From there I calculated what the potential would be for an infinite sum of masses (as opposed to the solid line of mass when $R >> L$). In this case, the distance from the test mass to each mirror image in the extra dimension is $R = \sqrt{((nL)^2 + r^2)}$, where r is the three dimensional distance, so the potential comes out to be:

$$\begin{aligned} V &= \frac{G_N^{(3)} LMm}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{((nL)^2 + r^2)} && (10) \\ &= \frac{G_N^{(3)} Mm}{\pi L} \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + (\frac{r}{L})^2)}. && (11) \end{aligned}$$

Mathematica computed this infinite sum analytically, and for large enough $\frac{r}{L}$ (i.e. where $r >> L$), the sum is:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left(n^2 + (\frac{r}{L})^2\right)} = \frac{\pi L}{r}, \tag{12}$$

and so

$$V = \frac{G_N^{(3)} Mm}{\pi L} \frac{\pi L}{r} = \frac{G_N^{(3)} Mm}{r}, \tag{13}$$

the regular $\frac{1}{r}$ potential.

For all other r, I was able to graph the natural log of $rV(r)$ versus the natural log of $\frac{r}{L}$. In graphing my results for these calculations, I allowed all constants in the equation (G, M, and L) to equal 1 for simplicity's sake. Figure 3 is the graph of the natural log of the gravitational potential versus the natural log of the ratio of the distance to the size of the dimension for 1 extra dimension.

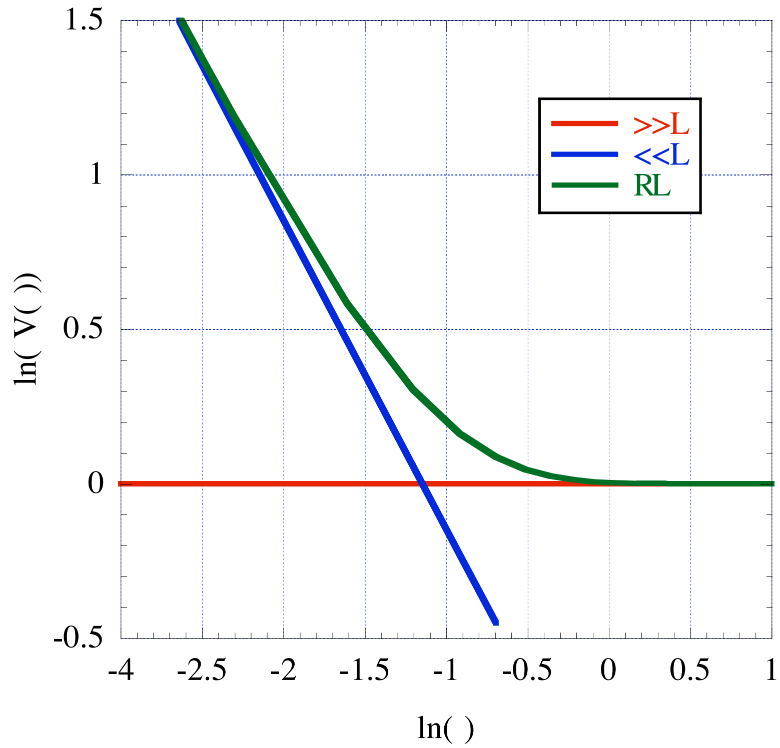

Figure 3: The gravitational potential times r is plotted as a function of r for one extra compact dimension. The blue line represents the potential already calculated for $r << L$, the red line is the $1/r$ potential for $r >> L$, and the green line is the calculation using the infinite sum.

The x-axis is the natural log of the distance, while the y-axis is the natural log of $r * V(r)$. You can see that for small and large values of r, the graph quickly converges to the respective potentials of $\frac{1}{r^2}$ for small r and $\frac{1}{r}$ for large r.

4 Two Extra Dimensions

For two extra dimensions, the shape of the dimensions also contributes to the potential. Instead of having a cylinder, we now have a two-torus, with different circumferences and different shape angles possible (Figure 4).

First, I calculated the potential for extra dimensions with the same length, then

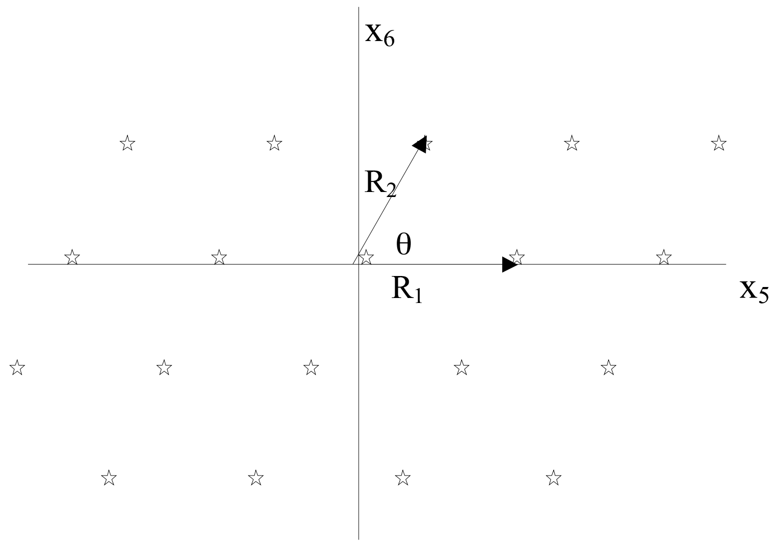

Figure 4: In two extra dimensions, the potential can be a function of both size and shape angle. The stars represent images of a point mass in a space with two compact dimensions- each distance L travelled in the direction of the extra dimension leads you back to the place you started, so that one object appears as an infinite plane of objects.

moved on to different radii and from there to different shape angles. The potential equation for $R_1 = R_2$ can be calculated from the 5-dimensional Gauss' law:

$$V = \frac{G_N^{(3)} L^2 M m}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(L^2(n^2+m^2)+r^2)^{3/2}} \tag{14}$$

$$= \frac{G_N^{(3)} M m}{2\pi L} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(n^2+m^2+(\frac{r}{L})^2)^{3/2}}. \tag{15}$$

Though this sum certainly converges, Mathematica could not analytically calculate it, so I instead created a do loop to 'manually' calculate the sum. The do loop was implemented with the following Mathematica command:

$$s = 0; Do[s = s + 1./(x^2 + y^2 + i^2)^{(3/2)}, x, 1, 1000, y, 1, 1000]; s$$

In doing this I was able to calculate the sum out to $m, n = 1,000$. Beyond $m, n =$

1000, it was possible to use an integral to approximate the sum out to a further limit. The integral approximation is as follows to calculate the error between $m, n = 1000$ and $m, n = 10000$:

$$error = \int_{x=1}^{10,000} \int_{y=1}^{10,000} \frac{1}{(x^2 + y^2 + (\frac{r}{L})^2)^{3/2}} dxdy - \int_{x=1}^{1,000} \int_{y=1}^{1,000} \frac{1}{(x^2 + y^2 + (\frac{r}{L})^2)^{3/2}} dxdy \tag{16}$$

Also, since I only made the do loop go from $m, n = 1, \ldots 1000$, for the complete potential I had to use the following equation:

$$\frac{G_5 M m}{2\pi L} \left[4 \sum_{n=1}^{10000} \sum_{m=1}^{10000} \frac{1}{(n^2 + m^2 + (\frac{r}{L})^2)^{3/2}} + 2 \sum_{-\infty}^{\infty} \frac{1}{(n^2 + r^2)^{3/2}} - \left(\frac{L}{r} \right)^3 \right], \tag{17}$$

where I multiplied the double sum by 4 to account for 4 quadrants, added the infinite lines where m or $n = 0$, and then subtracted the point where m and n both equal 0 to keep from double counting (see figure 5).

I then exported all of my data to kaleidagraph, where I created a natural log plot of the potentials I had found. The calculations for the 5-dimensional case were quite similar to that of the 4-dimensional case, and again, I allowed all constants to equal one. The graph of the natural log of the potential versus the natural log of the distance is shown in Figure 6.

Once again, you can see that the sum quickly approaches the 5-dimensional law at small distances, and approaches the standard $\frac{1}{r^2}$ law at large distances. If one were to compare Figure 6 to Figure 3 for one extra dimension, one would see that the potential indeed increases more rapidly as r decreases.

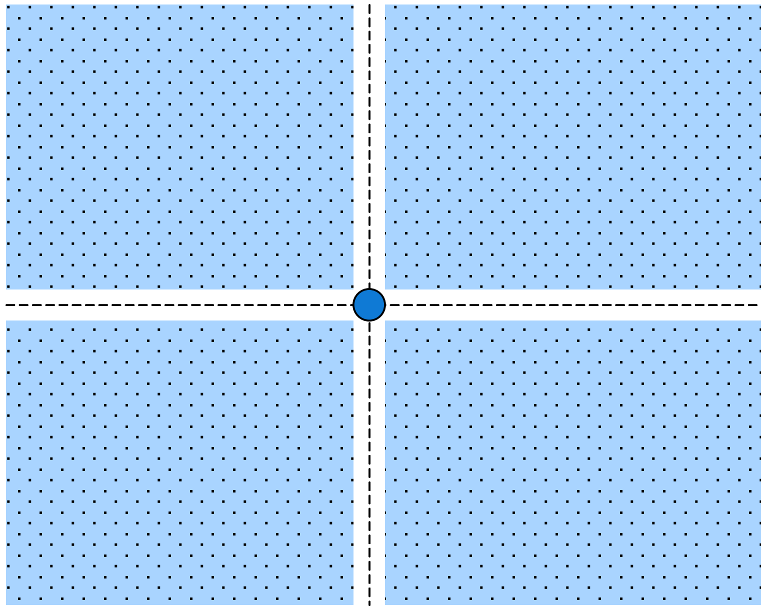

Figure 5: The sum from $m, n = 1, \ldots 1000$ must be multiplied by four for each quadrant, with the lines where $m = 0$ and $n = 0$ added in, and then the origin subtracted to keep from double counting.

4.1 Change in Length

There is no particular reason why the radii of two extra dimensions should be equal, so my next task was to calculate the potential for extra dimensions with different radii. For comparison with dimensions of equal size, I chose to calculate the potentials for $R_2 = 3R_1 = 3L$ and $R_2 = 10R_1 = 10L$. For simplicity, I will only go through the calculations for the case where $R_2 = 10R_1$, and only provide the results of my calculations for $R_2 = 3R_1$.

The first thing to note when changing the proportions of these dimensions is that this changes the volume V_n of the extra dimensions, and hence the five-dimensional Newton's constant changes as well. For $R_2 = 10R_1$, $G_N^{(5)}$ becomes:

$$G_N^{(5)} = \frac{3G_N^{(3)}V}{2\pi} = \frac{3G_N^{(3)}(10L^2)}{2\pi} = \frac{15G_N^{(3)}L^2}{\pi}. \tag{18}$$

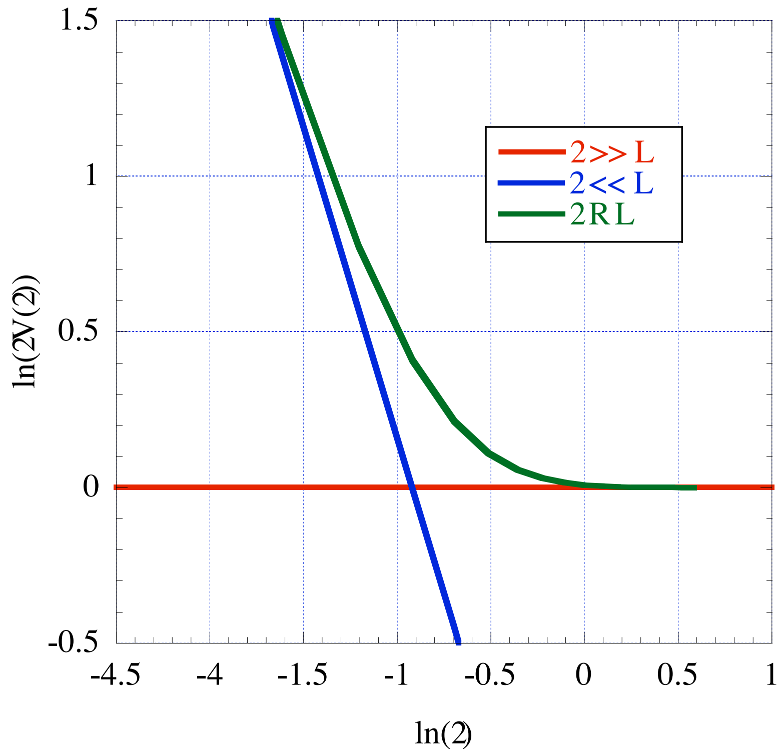

Figure 6: The gravitational potential times r plotted as a function of distance for two extra dimensions. The blue line is the calculated potential for distances sufficiently smaller than the size of the extra dimensions, and the $y = 0$ line is the potential for large distances.

Now that we have the new 5-dimensional constant, the process is quite similar to the calculations with $R_1 = R_2$. The force with the new constant is $F = \frac{G_N^{(5)} M}{r^4} = \frac{15 G_N^{(3)} L^2 M}{\pi r^4}$, and by integrating, we find that the five-dimensional potential is $V = \frac{5GL^2M}{r^3}$, where now the distance to each individual image is $R = \sqrt{r^2 + (nL)^2 + 100(mL)^2}$, where $m, n \in \mathbb{Z}$. So the full potential, summing over mirror images, is

$$V = \frac{5GM}{\pi L} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{(n^2 + 100m^2 + (\frac{r}{L})^2)^{3/2}}. \tag{19}$$

To calculate the potential using Mathematica, a similar process to the $R_1 = R_2$ case was necessary. I calculated the sums each from 1 to $1,000$, multiplied that sum by 4 to represent the four quadrants, and added in the zero lines. The only real difference was that this time the $m = 0$ and $n = 0$ lines required different sums. The final calculation of the potential was:

$$V = \frac{5GM}{\pi L}\left[4\sum_{1}^{1{,}000}\sum_{1}^{1{,}000}\frac{1}{(n^2+100m^2+(\frac{r}{L})^2)^{3/2}} + \sum_{-\infty}^{\infty}\frac{1}{(n^2+(\frac{r}{L})^2)^{3/2}} + \sum_{-\infty}^{\infty}\frac{1}{(100n^2+(\frac{r}{L})^2)^{3/2}} - \left(\frac{L}{r}\right)^3\right]. \quad (20)$$

By the same calculation, the final potential for $R_2 = 3R_1 = 3L$ was:

$$V = \frac{3GM}{2\pi L}\left[4\sum_{1}^{1{,}000}\sum_{1}^{1{,}000}\frac{1}{(n^2+9m^2+(\frac{r}{L})^2)^{3/2}} + \sum_{-\infty}^{\infty}\frac{1}{(n^2+(\frac{r}{L})^2)^{3/2}} + \sum_{-\infty}^{\infty}\frac{1}{(9n^2+(\frac{r}{L})^2)^{3/2}} - \left(\frac{L}{r}\right)^3\right]. \quad (21)$$

The results of these calculations are plotted on the graph in Figure 7, along with the potential for $R_1 = R_2$ for comparison.

Since the volumes of the extra dimensions change for each set of radii, the corresponding potential at high energies changes by a constant factor, so that instead of converging to the same line, each set of conditions approach a different line, all parallel to each other.

An important thing to note is that even though a change in shape of the dimensions led to a change in the potential at high energy, there is no way to tell simply from the volume of the dimensions what the shape is. $R_1 = R_2 = \sqrt{10}L$ would have the same potential as $R_2 = 10R_1 = L$. Figure 8 is the result of rescaling the volumes of the extra dimensions with differing radii, i.e. letting $R_2 = 10R_1 = \frac{L}{\sqrt{10}}$ and $R_2 = 3R_1 = \frac{L}{\sqrt{3}}$, so that the volume stays the same for each set of conditions. In this graph, one can see that the value of the potential between $R_2 = R_1$ and $R_2 = 10R_1$ varies by at most around a factor of two.

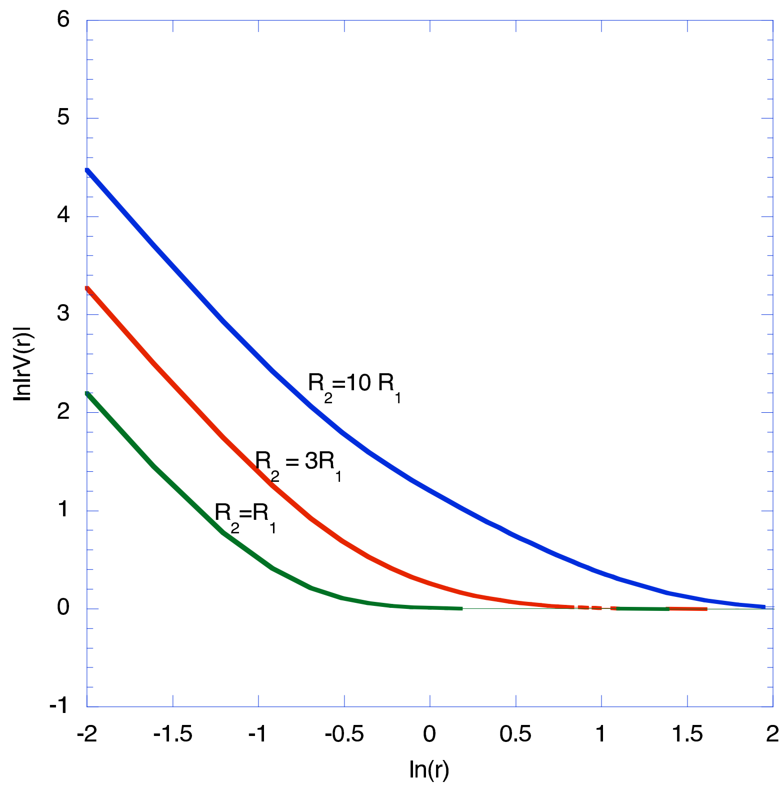

Figure 7: The gravitational potential times r plotted as function of distance for two extra dimensions with varying length ratios. Plotted are the cases where $R_2 = R_1$, $R_2 = 10R_1$, and $R_2 = 3R_1$.

4.2 Change in Shape Angle

Another possibility in the case of two extra dimenions is a change in shape angle. Depending on how big the angle θ between the radii is, the potential at intermediate distances changes. In this case, I computed the potentials for shape angles of $5°$ and $30°$. Again, I will only go through the calculations for one case ($30°$), and simply provide the results of the calculations for the other ($5°$).

The first step in calculating the potential for different shape angles, as it was for different radii, is to calculate the volume V_n of the extra dimensions and use that to calculate $G_N^{(5)}$, the force, and the potential. The volume of the extra dimensions in

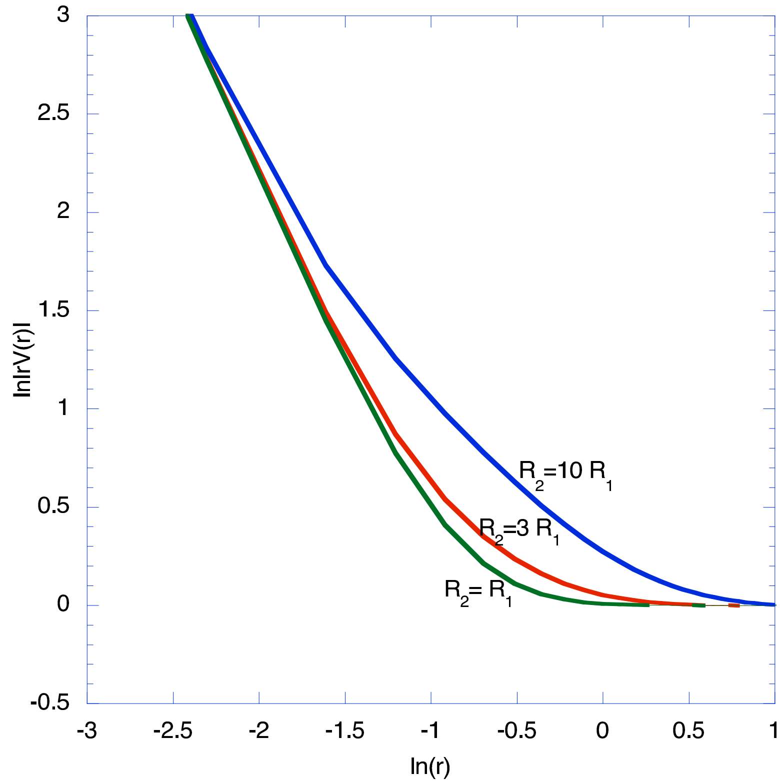

Figure 8: The graph of the 5-dimensional potential times r, as a function of distance for differing radii of two extra dimensions. In this case, all volumes were held constant so that a comparison between $R_2 = 10R_1$, $R_2 = 3R_1$, and $R_2 = R_1$ could be seen more clearly.

this case is the volume of a parallelogram with equal sides,

$$V_n = L(L\sin(30)) = \frac{L^2}{2} \implies G_N^{(5)} = \frac{3GV_n}{2\pi} = \frac{3G_N^{(3)}L^2}{4\pi}. \tag{22}$$

The distance from the origin to each point in the extra dimensions can be seen by referring to Figure 9. One can see that for any given m and n, the distance to that point will be

$$R = \sqrt{(mL + nL\cos\theta)^2 + (nL\cos\theta)^2 + r^2} \quad (23)$$

$$= \sqrt{L^2(m^2 + 2mn\cos\theta + n^2\cos^2\theta + n^2\sin^2\theta) + r^2} \quad (24)$$

$$= \sqrt{L^2(m^2 + n^2 + 2mn\cos\theta) + r^2}, \quad (25)$$

where r is the 3-dimensional distance.

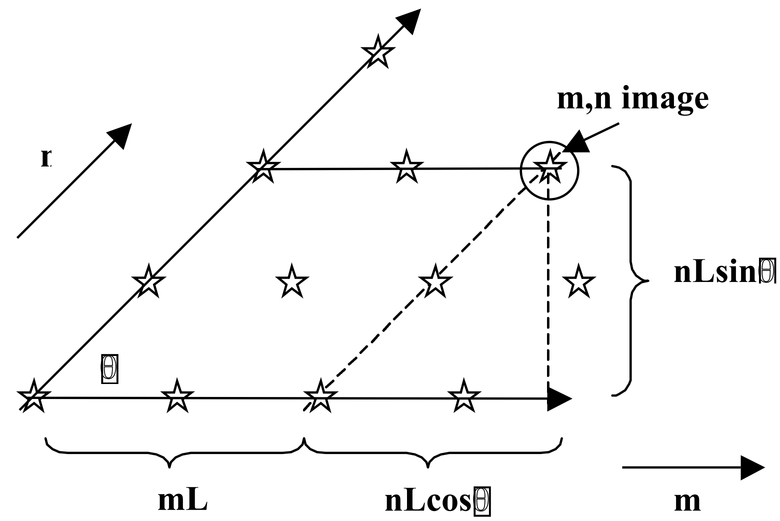

Figure 9: The trigonometry of two extra dimensions with a shape angle θ. In the x-direction, an individual mass excitation will be located $mL + nL\cos\theta$ from the origin, and in the y-direction the image will be $nL\sin\theta$ from the origin.

Thus for the case where the shape angle is 30°, $\cos\theta = \sqrt{3}/2$, the distance is $R = \sqrt{L^2(m^2 + n^2 + \sqrt{3}mn) + r^2}$, and so the potential is:

$$V = \frac{G_N^{(3)} M L^2}{4\pi} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{(L^2(n^2 + m^2) + \sqrt{3}mn + r^2)^{3/2}} \quad (26)$$

$$= \frac{G_N^{(3)} M}{4\pi L} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{(n^2 + m^2 + \sqrt{3}mn + (\frac{r}{L})^2)^{3/2}}. \quad (27)$$

For the case where $\theta = 5°$, by the equation for R, the distance between the origin and each point in the extra dimensions is $R = \sqrt{L^2(m^2 + n^2 + 2mn\cos\frac{\pi}{36}) + r^2}$, and the potential is

$$V = \frac{G_N^{(3)} M \sin\frac{\pi L}{36}}{2\pi} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{(n^2 + m^2 + 2mn\cos\frac{\pi}{36} + (\frac{r}{L})^2)^{3/2}}. \tag{28}$$

Unfortunately, since the dimensions are not orthogonal to each other, I cannot simply calculate the sums for $n = 1, \ldots 1000$ and multiply. Instead, I just had to calculate the sums for $m, n = -1000, \ldots, 1000$ and calculate any error using the integral approximation. Figure 10 shows the potentials for $\theta = \pi/2$, $\pi/6$ and $\pi/36$, where the volumes change in according to the shape angle. Figure 11 shows the potentials where, similar to the case with different radii, the volume is held fixed. One can see that the difference between $\theta = \pi/6$ and $\theta = \pi/2$ is nearly indistinguishable, while there is approximately a factor of two between $\theta = \pi/36$ and $\theta = \pi/2$.

5 Three Extra Dimensions

Now we move on to the possibility where there are three large extra dimensions. The mathematical method to calculate potentials using three extra dimensions follows the same outline as calculations for two extra dimensions: find the volume of the dimensions, calculate $G_N^{(6)}$, calculate the force, and finally integrate to find the potential. The difference comes in the time it takes for a computer to do such calculations, and the amount of geometry used to compensate for not summing over negative values.

Recall from the section on Gauss' Law that

$$G_N^{(3+n)} = \frac{4\pi V_n G_N^{(3)}}{2\pi^{(3+n)/2}/\Gamma(n/2)}, \tag{29}$$

which for three extra dimensions is

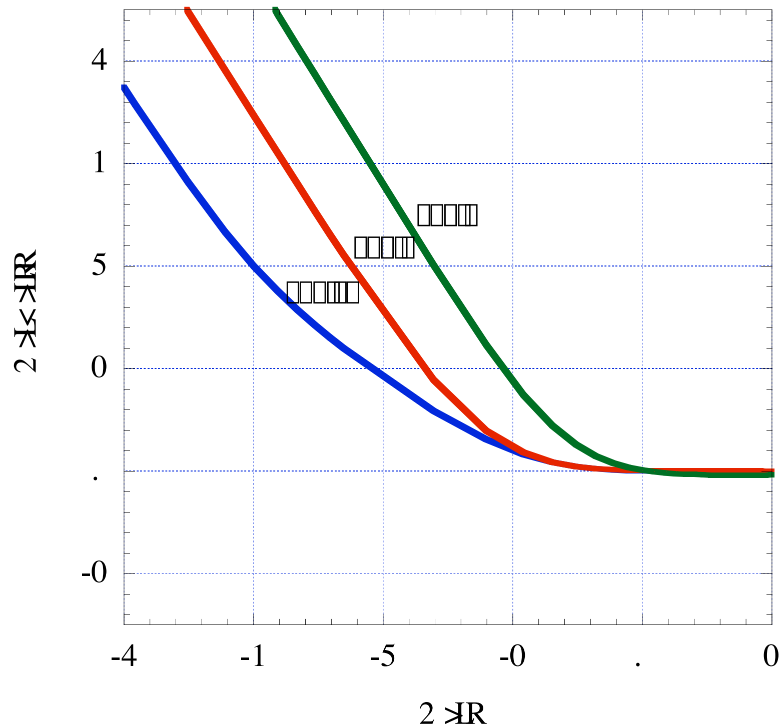

Figure 10: The graph of gravitational potential, times r, as a function of distance is plotted for two extra dimensions that are not orthogonal to each other. The shape angles shown are $\pi/36$, $\pi/6$, and $\pi/2$.

$$G_N^{(6)} = \frac{4\pi L^3 G_N^{(3)}}{2\pi^3/2!} = \frac{4L^3 G_N^{(3)}}{\pi^2}. \tag{30}$$

Using this value for $G_N^{(6)}$, we find that the force at small distances is $F = \frac{4G_N^{(3)} M L^3}{\pi^2 r^5}$, and by integrating the force, that the potential is

$$V = \frac{G_N^{(3)} M L^3}{\pi^2 r^4}. \tag{31}$$

For intermediate distances, the potential is found by summing over all of the mass images in the extra dimensions, which this time are distributed over a volume rather than a plane or a line. Hence the potential becomes:

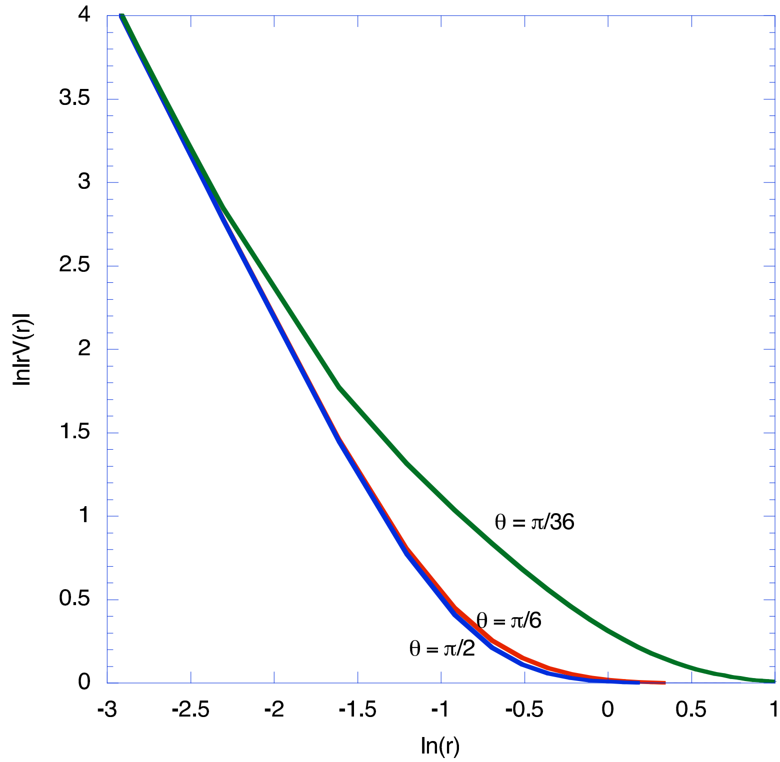

Figure 11: The graph of gravitational potential times r, plotted as a function of distance for two extra dimensions with shape angles of $\pi/36$, $\pi/6$, and $\pi/2$. This time the volumes of for each case were held the same, allowing the radius of each set of extra dimensions to vary instead.

$$V = \frac{G_N^{(3)} M}{\pi^2 L} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{(n^2 + m^2 + k^2 + (\frac{r}{L})^2)^2}. \tag{32}$$

In the process of computing the sum using Mathematica, I took m, n, and k each from 1 to 100, because the time taken to compute the triple sum out to points any further took too much time to be reasonable. An integral approximation similar to that in the 5-dimensional case was used, though for the purposes of graphing the data 100 was a large enough value for each iterator. Because I only computed the results for one octant of the total space, I had to multiply each result by 8 since the potential for each octant is identical, then add in the $m = 0$,$n = 0$ and $k = 0$ planes, subtract the $m, n = 0$, $n, k = 0$ and $m, k = 0$ lines to prevent double counting, and finally add back in the origin (Figure 12).

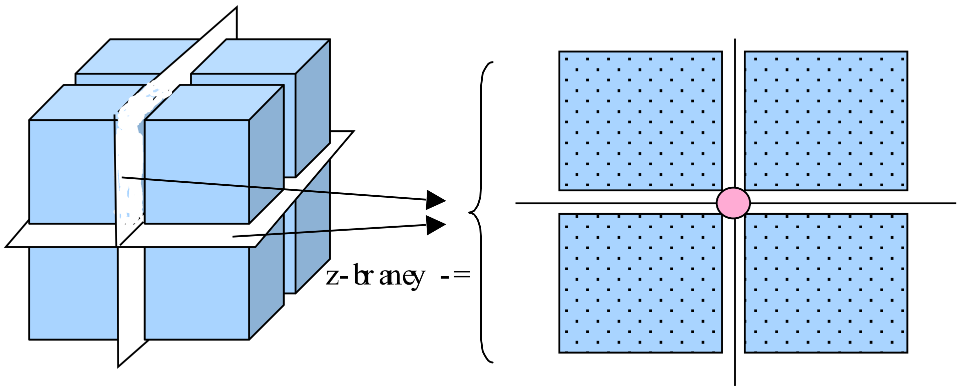

Figure 12: To find the potential in three extra dimensions, the triple sum is multiplied by 8, the zero-planes are added in, the zero-lines subtracted, and finally the origin added back in.

Thus the equation for the potential is as follows:

$$V = \frac{G_N^{(3)} M}{\pi^2 L} \Bigg[(8 \sum_{m=1}^{100} \sum_{n=1}^{100} \sum_{k=1}^{100} \frac{1}{(n^2 + m^2 + k^2 + (\frac{r}{L})^2)^2} + 3 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(n^2 + m^2 + (\frac{r}{L})^2)^2} - 3 \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + (\frac{r}{L})^2)^2} + \left(\frac{L}{r}\right)^4 \Bigg] \quad (33)$$

Now, as Figure 10 also shows, each zero-plane infinite sum must also be calculated from $m, n = 1 \ldots 1000$, each plane sum must be configured to compensate for only calculating one quadrant, so that the potential becomes:

$$V = \frac{G_N^{(3)} M}{\pi^2 L}\Bigg[8 \sum_{m=1}^{100}\sum_{n=1}^{100}\sum_{k=1}^{100} \frac{1}{(n^2+m^2+k^2+(\frac{r}{L})^2)^2} + 3\Bigg(4\sum_{n=1}^{1000}\sum_{m=1}^{1000}\frac{1}{(n^2+m^2+(\frac{r}{L})^2)^2} + 2\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+(\frac{r}{L})^2)^2} - \left(\frac{L}{r}\right)^4\Bigg) - 3\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+(\frac{r}{L})^2)^2} + \left(\frac{L}{r}\right)^4\Bigg] \tag{34}$$

$$= \frac{GM}{\pi^2 L}\Bigg[8 \sum_{m=1}^{100}\sum_{n=1}^{100}\sum_{k=1}^{100} \frac{1}{(n^2+m^2+k^2+(\frac{r}{L})^2)^2} + 12\sum_{n=1}^{1000}\sum_{m=1}^{1000}\frac{1}{(n^2+m^2+(\frac{r}{L})^2)^2} + 3\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+(\frac{r}{L})^2)^2} - 2\left(\frac{L}{r}\right)^4\Bigg] \tag{35}$$

The graph of the natural log of the potential in 6 dimensions is shown in Figure 13:

6 Compactification on a Sphere

To look at topologies that are no longer flat, such as a sphere, it is no longer possible to use the method of images that I have used previously. For example, on a sphere, one can return to one's starting point going in *any* direction. To calculate the potential on a sphere of two extra dimensions, we have to use the Yukawa potential rather than Gauss' Law.

With 2 extra spherical dimensions, the distance from one point to another will be the distance in 4-dimensional flat space plus the distance in spherical coordinates traversed on the surface of a sphere of radius R. Thus the metric, or the square of a differential path length, is:

$$ds^2 = dx_3^2 + R^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{36}$$

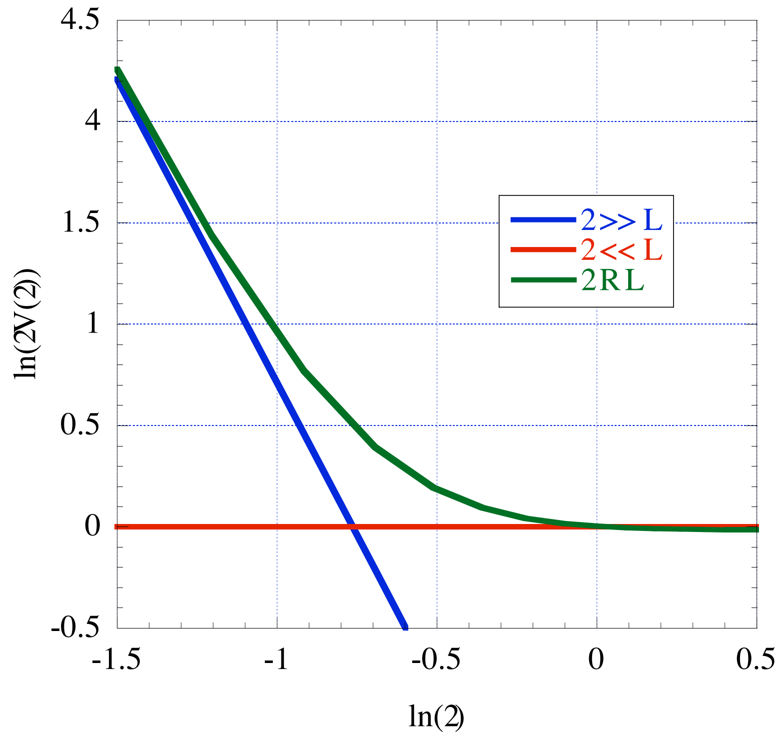

Figure 13: Natural log of the gravitational potential, times r, is plotted as a function of distance for three extra dimensions. The blue line is the $1/r^4$ potential expected for distances sufficiently smaller than the radii of the dimensions, and the red line is the standard $1/r$ potential for no extra dimensions.

Note that θ and ϕ are coordinates that extend outside the brane in 5 dimensions; for convenience we will set the 3-brane (the one we live on) at $\theta, \phi = 0$. We will use this metric to solve Poisson's equation for the Newtonian potential, keeping in mind that the Laplacian is the sum of the 3-dimensional Laplacian in cartesian coordinates plus the Laplacian in spherical coordinates for the two extra dimensions:

$$\nabla_5^2 V = 4\pi G_N \rho \implies \nabla_3^2 V + \nabla_S^2 V = 4\pi G_N \rho, \tag{37}$$

where ∇_S^2 can be found using the metric:

$$\nabla^2 V = \frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\partial^\mu V = \frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} g^{\mu\nu}\partial_\nu V \quad (38)$$

$$= \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}(\frac{R^2 \sin\theta}{R^2}\frac{\partial V}{\partial\theta}) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\phi}(\frac{R^2 \sin\theta}{R^2 \sin\theta^2}\frac{\partial V}{\partial\phi}) \quad (39)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{R^2 \sin\theta^2}\frac{\partial^2 V}{\partial\phi^2}, \quad (40)$$

where $\sqrt{g}$ is the square root of the determinant of the metric. To find our potential, we look for the Green's function, the solution to Poisson's equation with a delta function source:

$$\nabla_3^2 G + \nabla_S^2 G = \delta(\mathbf{x} - \mathbf{x}')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi'). \quad (41)$$

Akin to quantum mechanics, we want to find eigenvalue solutions to this equation:

$$(\nabla_3^2 + \nabla_S^2)\psi_{lm}^{\mathbf{k}} = E_{lm}^{\mathbf{k}}\psi_{lm}^{\mathbf{k}} \quad (42)$$

Like the eigenfunctions found in quantum mechanics, these ψ_{lm} follow the completeness relation:

$$\sum_{l,m} \psi_{lm}(x,\theta,\phi)\psi_{lm}^*(x',\theta',\phi') = \delta(x - x')\delta(\theta - \theta')\delta(\phi - \phi'), \quad (43)$$

so that

$$(\nabla_3^2 + \nabla_S^2)\frac{\sum_{l,m}\psi_{lm}\psi_{lm}^*}{E_{lm}} = \frac{E_{lm}\sum_{l,m}\psi_{lm}(x,\theta,\phi)\psi_{lm}^*(x,\theta,\phi)}{E_{lm}} = \delta(x - x')\delta(\theta - \theta')\delta(\phi - \phi'). \quad (44)$$

Thus, the Green's function G is the sum over all possible states of ψ_{lm}, which includes an integration:

$$G(\mathbf{x},\theta,\phi;\mathbf{x}',\theta',\phi') = \int \frac{d^3k}{(2\pi)^3} \sum_{l,m} \frac{\psi^{\mathbf{k}}_{lm}(\theta,\phi,\mathbf{x})\psi^{\mathbf{k}*}_{lm}(\theta',\phi',\mathbf{x}')}{E^{\mathbf{k}}_{lm}}, \tag{45}$$

where θ and ϕ are the variables of the 2 extra dimensions. Plugging this back into Eq. (41), in other words, multiplying the above equation by $E^{\mathbf{k}}_{lm}$, we get that

$$(\nabla^2_3 + \nabla^2_S)G = \int \frac{d^3k}{(2\pi)^3} \sum_{l,m} \psi^{\mathbf{k}}_{lm}(\theta,\phi,\mathbf{x})\psi^{\mathbf{k}*}_{lm}(\theta',\phi',\mathbf{x}'). \tag{46}$$

Now, using separation of variables, we can assume a solution $\psi^{\mathbf{k}}_{lm}$ that consists of a function dependent only on $\mathbf{x}$ and a function that depends only on θ and ϕ:

$$\psi^{\mathbf{k}}_{lm}(\mathbf{x},\theta,\phi) = \psi_{\mathbf{k}}(\mathbf{x})Y_{lm}(\theta,\phi). \tag{47}$$

If we take Eq. (42) and divide both sides by $\psi^{\mathbf{k}}_{lm}$, keeping in mind that $\psi_{\mathbf{k}}$ is dependent on the three normal dimensions, and Y_{lm} is dependent on the two spherical extra dimensions, we get the following equation:

$$\frac{\nabla^2_3\psi_{\mathbf{k}}(\mathbf{x})}{\psi_{\mathbf{k}}(\mathbf{x})} + \frac{\nabla^2_S Y_{lm}(\theta,\phi)}{Y_{lm}(\theta,\phi)} = E^{\mathbf{k}}_{lm}. \tag{48}$$

Because $\psi_{\mathbf{k}}(\mathbf{x})$ and $Y_{lm}(\theta,\phi)$ are by the nature of their dependent variables independent of each other, the above equation shows that each of the functions must equal a constant (if the sum of the two equals a constant and they are independent functions, they must both equal a constant). So now the process is to solve for each of the separated solutions:

$$\nabla^2\psi_{\mathbf{k}} = -k^2\psi_{\mathbf{k}} \implies \psi_{\mathbf{k}} = Ne^{i\mathbf{k}\cdot\mathbf{x}} \tag{49}$$

$$\nabla^2_S Y_{lm}(\theta,\phi) = N_{lm}Y_{lm}(\theta,\phi). \tag{50}$$

Equation (49) is just a standard 1st order differential equation with the known sine and cosine solution; the second will also turn out to be a familiar form. To find Y_{lm}, we start by expanding out the Laplacian for spherical coordinates:

$$\frac{1}{R^2 \sin\theta} \frac{\partial}{\partial\theta} \frac{\partial Y_{lm}}{\partial\theta} + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 Y_{lm}}{\partial\phi^2} = N_{lm} Y_{lm}. \tag{51}$$

Using separation of variables again, we separate Y_{lm} into a function dependent only on θ and a function dependent only on ϕ:

$$Y_{lm} = \Theta(\theta)\Phi(\phi). \tag{52}$$

Now, if we multiply Eq. (51) by $\sin^2(\theta)$ and divide the equation by $\Theta(\theta)\Phi(\phi)$, we get

$$\frac{\frac{\sin\theta}{R^2} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial\Theta}{\partial\theta}}{\Theta} + \frac{\frac{1}{R^2} \frac{\partial^2\Phi}{\partial\phi^2}}{\Phi} = N_{lm} \sin^2\theta. \tag{53}$$

Again, because the functions of Θ and Φ are independent of each other, the portions of Eq. (53) dependent on θ must equal a constant and the portion dependent on ϕ must also be a constant. Thus we get that

$$\frac{\sin\theta}{R^2} \frac{\partial}{\partial\theta} \frac{\partial\Theta}{\partial\theta} - N_{lm} \sin^2\theta\Theta = m^2\Theta \tag{54}$$

$$\frac{1}{R^2} \frac{\partial^2\Phi}{\partial\phi^2} = -m^2\Phi. \tag{55}$$

Like the solution $\psi(\mathbf{x})$, the solution for Φ is a combination of sines and cosines:

$$\Phi = a_m \cos(mR\phi) + b_m \sin(mR\phi), \tag{56}$$

and because we are on a sphere, and hence $\Phi(0) = \Phi(2\pi)$, we get that $m = \frac{n}{R}$, where $n \in \mathbb{Z}$. As for $\Theta(\theta)$, when $N_{lm} = \frac{l(l+1)}{k^2}$, the solutions turn out to be associated Legendre polynomials:

$$\Theta_{lm} = P_l^m(\cos\theta), \tag{57}$$

where $l = 0, \ldots, \infty$ and $m = -l, \ldots, l$. Thus the full solution for Y_{lm} is

$$Y_{lm}(\theta, \phi) = e^{im\phi} P_l^m(\cos\theta) \tag{58}$$

So, plugging our solutions for $Y_{lm}(\theta, \phi)$ and $\psi_{\mathbf{k}}(x)$ into the Green's function, we get the solution

$$G(\mathbf{x}, \theta, \phi; \mathbf{x}', \theta', \phi') = \int \frac{d^3k}{(2\pi)^3} \sum_{l,m} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi')}{\mathbf{k}^2 + \frac{l(l+1)}{R^2}}, \tag{59}$$

and by doing a complex contour integral over k, we get the final potential:

$$V = 4\pi G_N^{(3)} \sum_{l,m} \frac{e^{-\sqrt{\frac{l(l+1)}{R^2}}|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi'). \tag{60}$$

The constant term in the exponent, $\sqrt{l(l+1)}/R^2$, are what are known as Kaluza Klein masses; they are essentially excitations of the mass of a particle. Note that if $|\mathbf{x} - \mathbf{x}'| >> \mathbf{R}$, only the $l = 0$ term contributes to the potential. This means that $m = 0$, and hence Y_{lm} simply becomes a constant, and we get our $\frac{1}{r}$ potential back.

Since we are doing calculations on the brane, we can set θ and ϕ both equal to 0, so that

$$Y_{lm}(\theta, \phi) = e^{im\phi} P_l^m(\cos\theta) = e^0 P_l^m(1) = 1. \tag{61}$$

Because Y_{lm} is constant for all l and m, and the rest of the potential is independent of m, instead of doing a double sum over both l and m, we can just recognize that for each l, the sum over m will simply add a constant factor of $(l + 1)$ to the sum over l. So the potential is now:

$$V = 4\pi G_N^{(3)} \sum_{l=0}^{\infty} \frac{(l+1)e^{-\sqrt{\frac{(l+1)}{R^2}}|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{62}$$

When I calculated this potential, it correctly converged to the $1/r$ potential at large distances, but for small distances, it converged to a $1/r^3$ potential with an unexplained factor of two. Further research into spherical calculations will have to be done to discover the origin of this factor of two. Figure 14 depicts my findings for the spherical calculation.

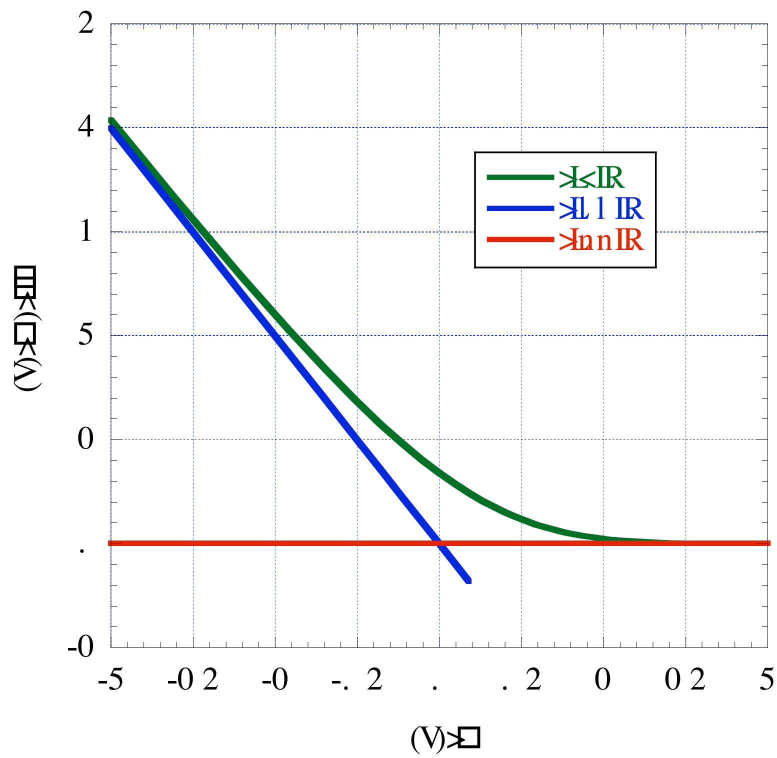

Figure 14: The gravitational potential, times r, plotted as a function of distance for two spherical extra dimensions. The blue line represents the $1/r^3$ potential at small distances, multiplied by an unexplained factor of two. Minus this factor of two, the potential correctly converges to predicted values at distances much greater than and much less than the size of the extra dimensions.

7 Conclusion

It is possible that in the next few years, both at the Large Hadron Collider and through other experiments, extra dimensions will be detected [3]. The functional form gravity would take is known at distances significantly smaller than the size of the dimensions, but my thesis is the first to explore the potential at distances at approximately the same size as the dimensions, as well as to show that constraints on the shape and volume of the extra dimensions can change the potential. One can see from my results that with two extra dimensions, it would be possible to detect a difference between a shape angle of 90 degrees and that of 5 degrees, and between the case where $R_1 = R_2$ and the case where the two radii are not the same. Though I touched on the idea of having extra dimensions that are not flat, by looking at spherical geometry, there is certainly the open possiblity of exploring other topologies.

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