Trajectories for a relativistic particle in a Kepler potential, Coulomb potential, or a scalar field

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Abstract

A recent paper by Timothy H. Boyer called "Unfamiliar trajectories for a relativistic particle in a Kepler or Coulomb potential" [1]contains new theoretical solutions and plots to equations of classical special relativistic trajectories (ignoring quantum effects) of a particle in a Kepler or Coulomb potential. After verifying and understanding Boyer's equations a computer is used to reproduce the same relativistic trajectories of a particle. The same computer plotting procedure is then used to analyze and plot solutions to relativistic trajectories in a Lorentz scalar field described in von Baeyer and C. M. Andersen's paper "On Classical Scalar Field Theories and the Relativistic Kepler Problem" [2]. All of this provides a better understanding of relativistic trajectories. These trajectories are important in understanding nuclear particle physics trajectories in relativistic theories.

1 Introduction

The purpose of this project is to learn about and become more familiar with classical special relativistic trajectories. Particles in nuclear particle physics obtain high velocities which changes the shape of observed trajectories due to special relativity. Boyer's paper shows that these special relativistic trajectories of particles in a Kepler or Coulomb potential can be theoretically analyzed and plotted [1]. By developing a computer technique to reproduce Boyer's trajectories, other important theoretical trajectories of particles in a Lorentz scalar field can be analyzed. These analyses will further the understanding of the effect of special relativity on particles in creating unfamiliar trajectories found in nature and nuclear particle physics.

2 Physics Background

Depending on the theory from which the equations of motion are derived, particles in a Coulomb or Kepler potential will undergo distinctly different types of trajectories. Non-relativistic particles attracted to a Coulomb are Kepler potential behave under Newton's equation of motion of $\frac{d}{dt}(m\vec{v}) = -\frac{\alpha}{r^2}\widehat{r}$. Here r is the distance of the particle from the center of the potential, α is the potential constant, m is the mass of the particle and v is the velocity. The trajectories are those described in classical Newtonian mechanics. They obey conservation of angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ and energy $E = \frac{1}{2}mv^2 - \frac{\alpha}{r}$. The particle rest energy mc^2 is not included in the energy equation. The focus of the paper is on special relativistic trajectories. Special relativistic particles that obey Lorentz vector transformations derived from electromagnetic theory have an equation of motion

$$\frac{d}{dt}(\gamma m\vec{v}) = -\frac{\alpha}{r^2}\widehat{r} \tag{1}$$

inside a Coulomb or Kepler potential. These particles still conserve their relativistic angular momentum $\vec{L} = \vec{r} \times \gamma m\vec{v}$ and energy $E = \gamma mc^2 - \frac{\alpha}{r}$ (including rest mass energy). Here γ is the familiar $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. In Bergmann theory the potential transforms as a Lorentz scalar. Special relativistic particles that obey Lorentz scalar transformations have theoretical trajectories that have an equation of motion

$$\frac{d\vec{p}}{dt} = -\frac{1}{\gamma}\frac{\alpha}{r^2}\widehat{r};\ \vec{p} = E\vec{v} \tag{2}$$

in a Coulomb or Kepler Potential. The particles also conserve relativistic angular momentum ($\vec{L} = \vec{r} \times E\vec{v}$) and energy ($E = \gamma mc^2 - \gamma\frac{\alpha}{r}$; including rest mass energy).

3 Geometry of Trajectories

Many of the special relativistic trajectories have an orbital solution of the form of the general equation

$$\frac{1}{r} = A + B\cos{(1+\epsilon)\theta} \tag{3}$$

By varying the parameters A, B, and ϵ, the path described by going θ degrees around the origin creates six closed orbits: circles, ellipses, precessing counter-clockwise ellipses, precessing clock-wise ellipses, elliptical lobes, and precessing elliptical lobes; and, two open orbits: parabola-shaped and loops.

Six closed orbits that do not scatter to infinity are created when $B < A$:

Closed orbit 1. A closed orbit with $\epsilon = -1$ or B = 0, describes a circle.

For elliptical orbits, $|B|$ determines the distance between the farthest part of the ellipse from the origin to the closest (eccentricity = perihelion - aphelion).

Closed orbit 2. A closed orbit with $\epsilon = 0$ describes an ellipse.

Closed orbit 3. A closed orbit with $\epsilon < 0$ describes a precessing ellipse. The particle goes counter-clockwise following an ellipse that precesses counter-clockwise.

Closed orbit 4. A closed orbit with $0 < \epsilon < 1$ describes an ellipse precessing in the other direction. The particle goes counter-clockwise, but due to the $(1+\epsilon) > 1$ inside the cosine of the general equation, the ellipse instead precesses clockwise. The rate of precession is proportional to ϵ.

Closed orbit 5. A closed orbit with $\epsilon = 1$ and ϵ an integer whole number describes an elliptical lobed pathway. The particle's pathway is sinusoidal counter-clockwise along an ellipse creating a multi-lobed-shape pathway. The pathway oscillates $(1 + \epsilon)$ times of full sinusoidal periods creating $(1 + \epsilon)$ lobes.
Closed orbit 6. A closed orbit with $\epsilon > 1$ and ϵ is not an integer whole number, describes an elliptical precessing multi-lobed pathway. The orbit also precesses clockwise due to the $(1+\epsilon) > 1$. For each revolution around the center, the pathway oscillates $(1+\epsilon)$ times of full sinusoidal periods creating $(1 + \epsilon)$ lobes. The rate of precession of the multi-lobe pathway is proportional to ϵ.

Two open orbits that scatter to infinity are described by the general equation when B = A: The range of angle θ the particle travels around the origin center is

$$\theta_r = \frac{2 \arccos\left(\frac{-A}{B}\right)}{1 + \epsilon} \tag{4}$$

Open orbit 1: When the range of $\theta_r < 2\pi$, the orbit describes a parabola-like pathway that scatters to infinity. For the range of theta to be less than that of a complete revolution, the ratio of $\frac{A}{B}$ must be small enough and $(1 + \epsilon)$ large enough so that the particle scatters as a parabola instead of looping all the way around the center.
Open orbit 2: An open orbit with a range of $\theta_r > 2\pi$, describes a parabola-shaped path that loops around the center $\frac{\theta_r}{2\pi}$ times. A third mathematical, but unrealistic orbit of a broken straight line when $\theta_r = 2\pi$ will not be studied. In general for all of the orbits: Increasing A in the equation decreases the size of the orbit. Decreasing A in the equation increases the size of the orbit.

4 Vector Theory

Boyer's work describes various particle special relativistic trajectories that transform to the Lorentz Vector transformations that we see in life [1]. Some of the trajectories are noticed in nature and some are theoretical cases where quantum effects are ignored.

Special relativistic trajectories that obey Lorentz vector transformations in a Coulomb or Kepler potential are studied and plotted by Boyer in [1]. Equations (37),(38),(39)

$$s = \frac{1}{r} = \sqrt{\frac{E^2L^2c^2 - m^2c^4(L^2c^2 - \alpha^2)}{(L^2c^2 - \alpha^2)^2}} \cos\left(\sqrt{1 - \left(\frac{\alpha}{Lc}\right)^2}(\theta - \theta_0)\right) + \frac{E\alpha}{L^2c^2 - \alpha^2} \tag{5}$$

$$s = \frac{1}{r} = \frac{1}{2}(\frac{E}{\alpha})(\theta - \theta_0)^2 + \frac{m^2c^4 - E^2}{2E\alpha} \tag{6}$$

$$s = \frac{1}{r} = \sqrt{\frac{m^2c^4(\alpha^2 - L^2c^2) + E^2L^2c^2}{(\alpha^2 - L^2c^2)^2}} \cosh\left(\sqrt{(\frac{\alpha}{Lc})^2 - 1}(\theta - \theta_0)\right) - \frac{E\alpha}{\alpha^2 - L^2c^2} \tag{7}$$

derived in his paper are the solutions to the equation of motion 1. By varying the amount of relativistic energy and angular momentum given to a particle, Maple or a TI calculator is used to reproduce plots of r (the distance from the center of the potential) vs. θ. Analysis of the types of trajectories will give an understanding of relativistic trajectories. Simplification of the trajectory solutions have been done by choosing units that are easier to work with. Conversion back to standard units will maintain the same trajectory shape. The units chosen are to make the Kepler or Coulomb potential constant alpha $\alpha = 1$, the mass of the particle m = 1, and the speed of light c = 1. In a coulomb field these units would represent: Mass of electron = 1. The classical electron radius = 1. Time for particle to travel the distance of a classical electron radius = 1.

The velocity of a particle has an upper limit equal to the speed of light c. This imposes a minimum angular momentum that a relativistic particle must have to keep a bound circular orbit. The minimum critical angular momentum

$$L_\alpha = \frac{\alpha}{c} = 1 \tag{8}$$

(Calculations in [1]) is a very important boundary between different types of orbits.

Qualitatively, significantly different trajectories are produced when the particle is given energy greater than or smaller than $E = mc^2 = 1$ and angular momentum above or below the critical angular momentum L_α. A summary of the different types of trajectories of particles with different energy E and angular momentum L is shown in the following Figure 1.

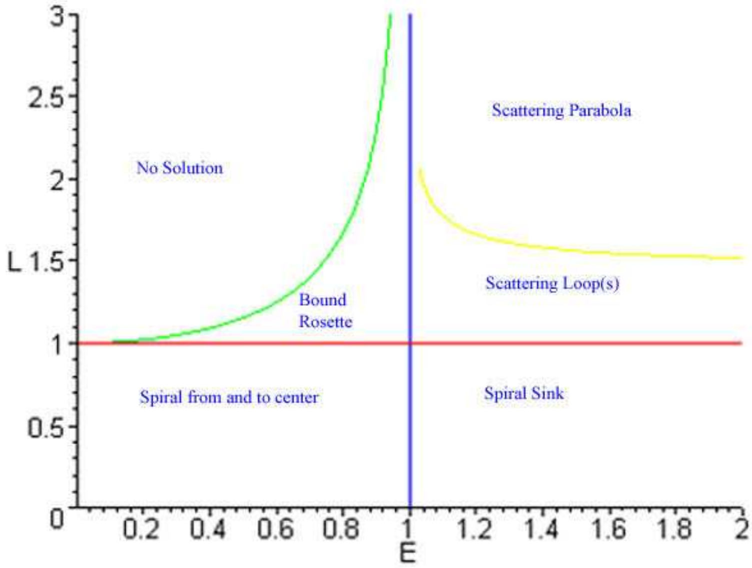

Figure 1: E vs. L plot for Vector Theory.

Physical and mathematical descriptions of the trajectories plotted by calculator and computer are covered in the rest of this section:

Particles with energy $E \geq mc^2 = 1$ have a greater kinetic energy than potential energy enabling them to scatter to infinity. For particle solutions that have the general form of 3, this is equivalent to setting B=A.

Special relativistic trajectories of particles with energy $E \geq 1$ and angular momentum $L >$ (a function of E) scatter similar to non-relativistic particles; they have the same solution as the parabola open orbit 1 discussed in the geometry section. In order to have a range of $\theta_r < 2\pi$, the particle's angular momentum must be large enough. With large angular momentum the particle is traveling faster perpendicular to the center of the attractive potential with farther distance so it is less likely to be pulled into a complete orbital loop. As shown in Figure 2, the particle comes in, goes around the attractive center, and then shoots away like an asteroid.

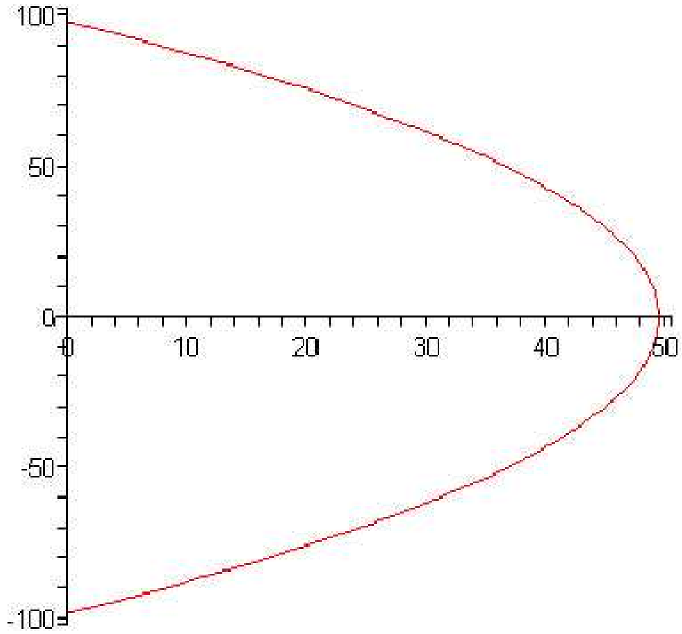

Figure 2: Scattering Parabola Trajectory, E=1 L=10 $E \geq 1$ large L.

(The trajectories in the figures are drawn for arbitrary ranges of theta for convenience to show the general characteristics of the trajectories.) Changing the initial starting position and velocity will cause the particle to come in and out from different directions.

When the energy $E > 1$ and angular momentum L is small but greater than L_α, the particle follows open orbit 2, a loop trajectory. The particle travels around the center for more than 360 degrees. This unfamiliar trajectory is unique to special relativity. As the particle spirals in toward the potential center the velocity increases, and due to special relativity this greatly increases the particle's relativistic mass, which decreases the acceleration toward the center allowing the particle to escape again. So the final trajectory looks like a scattering trajectory with one or more loops around the center as seen in Figure 3.

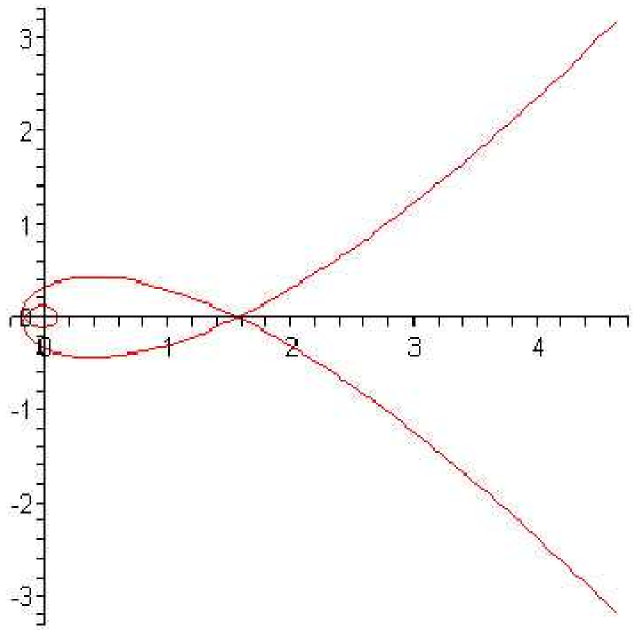

Figure 3: Scattering Loop Trajectory, E=1 L=1.1 $E \geq 1$ $L > E$ (slightly).

Non-relativistic particles that come in from far away with small non-zero angular momentum never reach the center of the potential, but scatter away as a parabola. However, if a relativistic particle is given sufficiently small angular momentum ($L \leq L_\alpha$: has orbital solution from eqs. 6 and 7), the particle will continue to spiral into the potential center as shown in Figure 4.

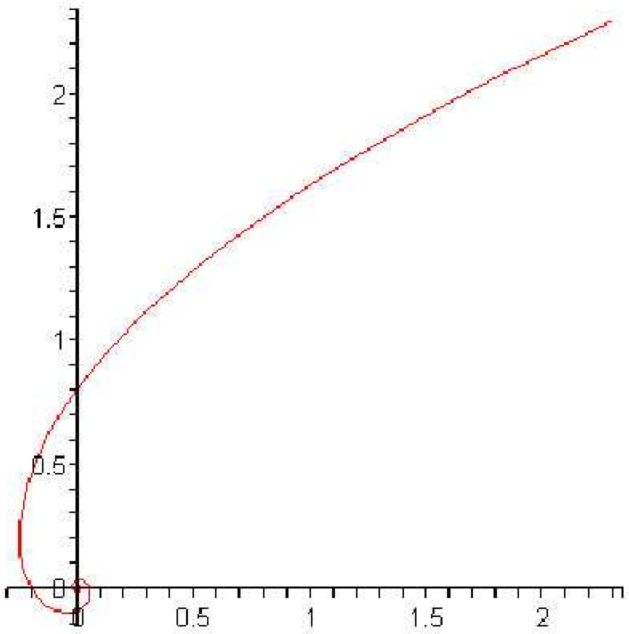

Figure 4: Spiral Sink Trajectory, E=1 L=1 $E \geq 1$ $L \leq 1$.

If instead the angular momentum is kept large and the total relativistic energy is made smaller than the particle's rest mass energy, the particle will orbit around the center, but the orbital trajectory will continue to change shape due to special relativity affecting the relativistic mass of the particle that is attracted at various points. The result is the same as closed orbit 3, an elliptical orbit that precesses around the center. The relativistic velocity causes time dilation, which affects the rate of perihelion advance or precession. The overall trajectory looks like many orbits that together form a rosette shape seen in Figure 5. The angular momentum is large enough to keep the radius of each orbital loop constant.

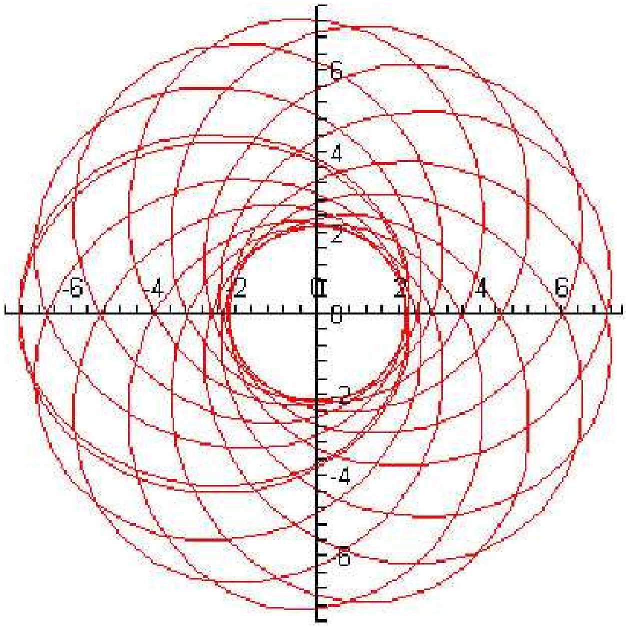

Figure 5: Rosette Trajectory, E=.9 L=2 $0 < E < 1$ $L > 1$.

When both the angular momentum and energy are small, the particle will have a bound orbit where it spirals away from and back in toward the potential center. Starting near the center, the particle's velocity is very large, and due to special relativity increases the relativistic mass so that the acceleration toward the center decreases enough that the particle spirals out. Since the particle's kinetic energy is less than its potential energy it will be pulled back in to complete the orbit, as shown in Figure 6.

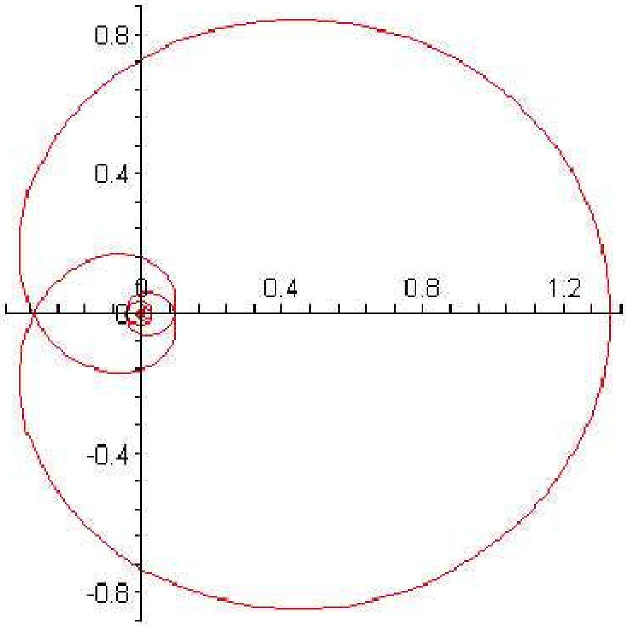

Figure 6: Spiral to and from center Trajectory, E=0.5 L=1 $E < 1$ $L \leq 1$.

5 Scalar Theory

Trajectories subject to Lorentz scalar transformations in a Coulomb or Kepler potential have the equation of motion 2 derived by Baeyer and C. M. Andersen [2] from Bergmann's theory.

The solution to the equation of motion is

$$s = \frac{1}{r} = \frac{\sqrt{\alpha^2 E^2 - L^2 m^2 + E^2 L^2}}{L^2 + \alpha^2} \cos \sqrt{1 + (\frac{\alpha}{Lc})^2}(\theta - \theta_0) + \frac{\alpha m}{L^2 + \alpha^2} \tag{9}$$

(The equation has been simplified by setting c = 1.) The same plotting technique used in vector theory is used to plot solutions of scalar field trajectories that have never been seen before.

As in vector fields, particles with differing amounts of energy and angular momentum will have qualitatively different types of trajectories in scalar fields. A grouping of particles with energy E and angular momentum L are shown in Figure 7.

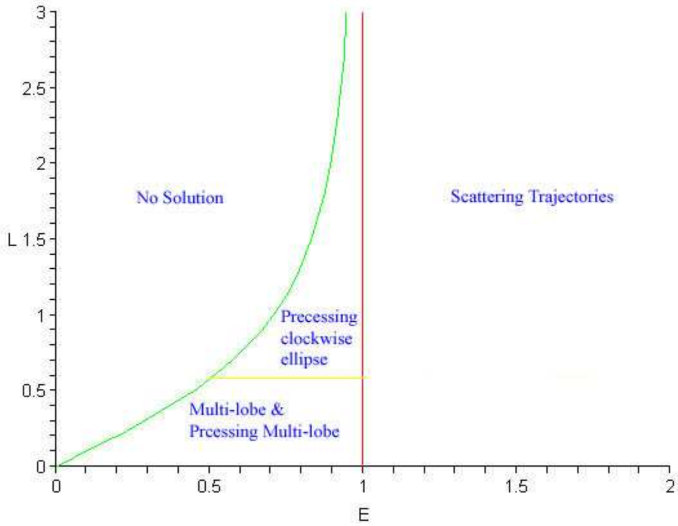

Figure 7: E vs. L plot for Scalar Theory.

Figure 7. E vs. L plot for Scalar Theory Since there is no cosh solution, the spiral sink and the spiral in and out from the center are found only in vector fields, not scalar fields. When a particle in a scalar field has an angular momentum $L > \frac{\alpha}{\sqrt{3}}$, it is equivalent to having a solution to the general orbit equation where $(1 + \epsilon) > 1$ describe by closed orbit 4. The elliptical orbit to precesses clockwise unlike the vector theory case where the ellipse precesses counter-clockwise. The rate of precession is proportional to $(1 + \epsilon)$, so the less angular momentum L the particle has, the faster it precesses. A sample orbit is shown in Figure 8.

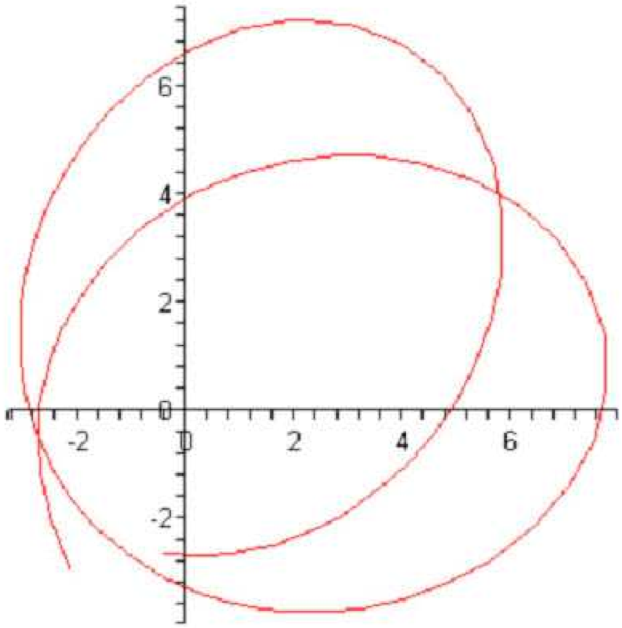

Figure 8: Precessing Clockwise Ellipse Trajectory(Rosette), E=0.9 L=1.727; $E < 1$ $L > \frac{\alpha}{\sqrt{3}}$.

In scalar fields with Coulomb or Kepler potentials it is also seen that the less angular momentum the particle has, the more it oscillates to and from the potential center. With sufficiently small angular momentum, one can see the creation of lobes as the particle travels around the center. When the particle has energy $E < 1$ and angular momentum $L = \frac{\alpha}{\sqrt{n^2-1}}$, where n=whole integer, a closed multi-lobed trajectory equivalent to closed orbit 5 is created; shown in Figure 9.

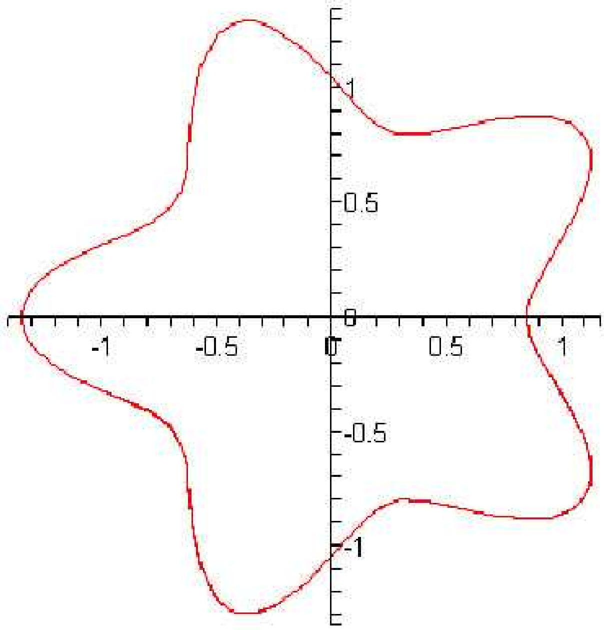

Figure 9: Multi-lobed Trajectory, E=.35 $L = \frac{1}{\sqrt{5^2-1}}$ $E < 1$ $L = \frac{\alpha}{\sqrt{n^2-1}}$.

When the particle has energy $E < 1$ and angular momentum $L < \frac{\alpha}{\sqrt{3}}$ and $L \neq \frac{\alpha}{\sqrt{n^2-1}}$, the particle has instead a precessing lobed trajectory (described in closed orbit 6) in the scalar field potential; shown in Figure 10.

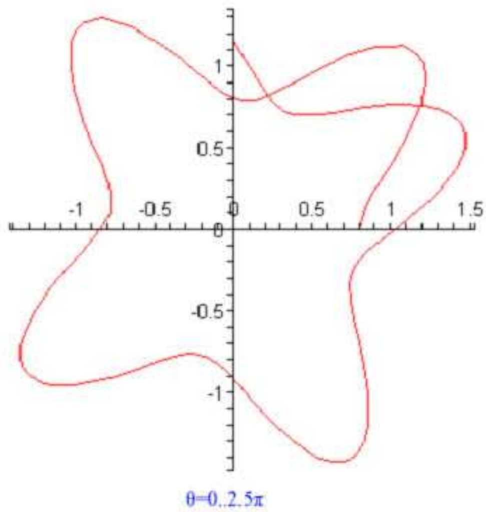

Figure 10: Precessing clock-wise multi-lobed Trajectory, E=0.4 L=0.0022 $E < 1$ $L \neq \frac{\alpha}{\sqrt{n^2-1}}$.

With enough revolutions around the center, the pathway looks similar to that of a rosette. More investigation is needed in analyzing scattering trajectories.

6 Conclusion

The modeling of special relativistic trajectories in Coulomb or Kepler potentials gives helpful qualitative insight to what the trajectories look like, so such trajectories will be easier to recognize in nature and be easier to analyze using comparison. It is interesting to see that depending on which relativistic transformation theory is used, significantly different particle trajectories will be observed. Although scalar theory trajectories are currently theoretical, they might not be in the future; and even if they aren't found, they're similar enough for comparison to other trajectories found in nature. Most remarkable are the scalar lobed trajectories that look like an electron orbiting an atom diagram. It's also interesting to note the contrast of the two theories; where one particle orbit is precessing in one direction, the other precesses in the opposite. With better understanding of these various special relativistic trajectories, we will be closer to answering how and why particles in nature and nuclear particle physics follow the trajectories that they do. In the future further analysis of scattering trajectories in scalar fields can be done. In addition, it'll be interesting to try to find similar trajectories in nature that have the same appearance as any of the relativistic trajectories, and then we'll be better able to analyze and determine the physical significance of them.

7 Acknowledgements

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References

[1] T. Boyer, Am. J. of Physics **72**, 992 (2004).

[2] C. M. Andersen and H. C. von Baeyer, Ann. Phys. **62**, 120 (1971).