Measuring the Possible Fractional Dimensionality of the Universe

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by

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Abstract

Current attempts at unification propose relaxing the requirement that the universe must have an integer value of dimensions for a universe with fractional dimensions. Given a mathematical model of the universe using fractional dimensions, certain issues that exist using a integer dimensional model do not appear; however, the integer dimensional model has proven quite successful in describing most of the phenomenon we see. In this paper, I review the various mathematical constructs necessary to describe a fractional dimensional universe and use these constructs to redefine necessary physical equations. Then I will study various solutions to these equations as attempts to discover ways to verify and measure the fractionality of the universe.

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Contents

1 Introduction

The inconsistencies between General Relativity and Quantum Theory have left the former with many unanswered questions, such as how does General Relativity relate to vacuum energies and quantum mechanics[1]. The search for a consistent theory, or quantum gravity, still remains and establishes doubt upon the validity of how we view the universe as a whole. Among the various doubts include the choice of mathematical coordinate systems[2] whereby currently we view space time as generally flat on a microscopic level and curved relative to local masses on a macroscopic level. Although valid on the macroscopic level, this system creates singularities within the field equations on a microscopic level. The various attempts to solve this problem include coordinate systems with fractal spaces which thereby eliminate these singularities [3].

2 Mathematics of a Multifractional Universe

To allow for multifractional time and space sets, one assigns to each time-space point a dimensionality characteristic of an equation $d_t(r(t), t)$ and $d_r(t(r), r)$. By point, one assumes an interval of space and time as it approaches singularity. By interval one assumes a member of a multifractal set with global dimensions. These dimensions then determine the Lagrangian energy densities for all physical fields within these points. Thus to establish any physical equation of an interaction within the framework of a multifractional time-space universe, one must take the original Lagrangian describing the interaction and solve the generalized Euler equations using a substitution for the derivatives that includes the multifractional aspects. Fortunately, the substitution for derivatives involves using the fractional Riemann-Liouville derivative definitions:

$$D^d_{+,t}f(t) = \left(\frac{d}{dt}\right)^n \int_a^t dt' \frac{f(t')}{\Gamma(n-d)(t-t')^{(d-n+1)}} \tag{1}$$

$$D^d_{-,t}f(t) = (-1)^n \left(\frac{d}{dt}\right)^n \int_t^b dt' \frac{f(t')}{\Gamma(n-d)(t'-t)^{(d-n+1)}} \tag{2}$$

Given these equations, a and b are stationary values on an infinite axis where $a < b$. We make these derivatives suitable for fractional dimensions by allowing $d(t_i) = d(t)$; or rather, we allow the dimensionality between intervals to depend on the dimension we are deriving $f(t)$ on and allow $d(t)$ to be continuous rather than discretely defined for each point in time-space. The results are similar:

$$D^{d_t}_{+,t} f(t) = \left(\frac{d}{dt}\right)^n \int_a^t dt' \frac{f(t')}{\Gamma(n - d_t(t'))(t - t')^{d_t(t') - n + 1}} \tag{3}$$

$$D^{d_t}_{-,t} f(t) = (-1)^n \left(\frac{d}{dt}\right)^n \int_t^b dt' \frac{f(t')}{\Gamma(n - d_t(t'))(t' - t)^{d_t(t') - n + 1}} \tag{4}$$

Given that $d_t = d_t(r, t)$, the derivative becomes solely dependent on t. Note that now $n - 1_t < n, n = \{d_t\} + 1$ where $\{d_t\}$ is the integer part of $d_t \leq 0$, and $n = 0$ for $d_t < 0$. Note also that it is possible to substitute d_t with $d_r(r, t)$ when taking a derivative over r thus allowing for $D^{d_r}_{+,r} f(r, t)$.

Given these equations, it is necessary to describe the fractal dimensions of time $(d_t(r(t), t))$ with regards to the physical fields, $(\Phi_i(r(t), t), i = 1, 2, ...)$ or to the Lagrangians L_i. Thus the fractional Riemann-Lioville derivatives acting within the generalized Euler-Lagrange equations creates a system of differential equations with multiple solutions for d_t. Following the argument proposed by Kobeyev[3]:

$$d_t(r(t), t) = 1 + \sum_i \beta_i L_i(\Phi_i(r(t), t)) \tag{5}$$

where L_i are densities of energy of physical fields and $\beta_i L_i$ is a dimensional constant with physical dimension $[L_i]^{-1}$ whose value is not limited unless such limitations are imposed on the corresponding Lagrangians[4]. Such solutions also allow for the substitution of ordinary derivatives with fractional derivatives in most situations. Situations in which a solution where derivative substitution is possible, it may be necessary to include in the equations terms that approach zero as $r \to 0$[6].

3 Modified Physical Equations

3.1 Newton's Equations

Thus Newton's equations modified are:

$$D_{-,t}^{d_t(r,t)} D_{+,t}^{d_t(r,t)} = D_{+,t}^{d_r} \Phi_g(r(t))$$

As previously mentioned, the inclusion of b_g is necessary for the Euler-Lagrange equations to have a solution and b_g^{-1} is on the order of the size of the universe.

3.2 Maxwell's Equations

Thus Maxwell's equations modified are:

$$\sum_{i=1}^{3} D_{-,i,r}^{d_r} D_{+,i,r}^{d_r} A_j(x) - \frac{1}{c^2} D_{-,t}^{d_t} D_{+,t}^{d_t} A_j(x) + m^2 A_j(x) = \frac{4\pi}{c} j_j(x) \tag{7}$$

$$D_{+,j,r}^{d_j} A_j(x) = 0 \tag{8}$$

where d_j is equal to d_r for $j = 1, 2, 3$ and d_t for $j = 0$. The mass of the photon was included to allow for a solution of the Euler-Lagrange equations.

3.3 Schrodinger's Equation

$$-ihD_{+,t}^{d_t} \Psi(r,t) = -\frac{h^2}{2m} D_{-,r}^{d_r} D_{+,r}^{d_r} \Psi(r,t) - e^2(r,t)\Psi(r,t) \tag{9}$$

3.4 Dirac Equation

Thus Dirac's equation modified is:

$$[i\gamma_i(D_{+,i}^{d_i} - ieA_i(x)) - m]\Psi(x) = 0 \tag{10}$$

where γ_i are the ordinary Dirac matrices.

4 Approximation Methods

Given d_α where n is an integer, one can substitute for $d_\alpha = 1 + \epsilon(r(t), t)$ for $\alpha = r, t$. Within this case, the fractional derivatives can be approximated to be:

$$D^{1+\epsilon}_{+,x_\alpha} f(r(t), t) = \frac{\partial}{\partial t} f(r(t), t)) + \frac{\partial}{\partial t}[\epsilon(r(t), t) f(r(t), t)] \tag{11}$$

Thus for Newton's equations, one can solve for small approximations by first defining d_t to be:

$$d_t = 1 + \beta_g \Phi_g + \beta_g \Phi_m \tag{12}$$

where Φ_g is the gravitational potential caused by mass M with regards to the center of mass m and Φ_m is the gravitational potential caused by mass m. By allowing $D^{d_r}_{+,r} \approx \bigtriangledown$ and by ignoring the terms arising from b_g, the equation of motion becomes:

$$[1 - 2\epsilon(r(t))] \frac{d^2}{dt^2}] r = F_g \tag{13}$$

where

$$F_g = - \bigtriangledown \frac{\gamma M}{r}, \epsilon = \beta_g(\Phi_g + \Phi_m) \tag{14}$$

We now allow β_g to equal c^{-2} for potentials and allow the body with mass m to move in the gravitation field of mass M on the distance r where r_0 is the gravitational radius of the body with mass M ($r >> r_0 \quad M >> m$). These assumptions allow us to view these masses as points and therefore:

$$\Phi_m = \frac{am\gamma}{r_m} \tag{15}$$

where a is a factor associated with the mass distribution, near unity for spheres. Given the solution for the first equation of motion, the second equation can be calculated to be:

$$[1 - \frac{2\gamma}{c^2 r}(1 + \frac{amr}{Mr_m})](\frac{\partial r(t)}{\partial t})^2 - \frac{2mc^2}{r} + [\frac{2\gamma}{c^2 r}(1 + \frac{amr}{Mr_m})] r^2 (\frac{\partial \varphi(t)}{\partial t})^2 = 2E \tag{16}$$

This equation is different from the classical limit of the equations of general relativity by the existence of the additional terms in the square brackets which is combined with the approximate form of the first Newton's equation

$$[\frac{d^2}{dt^2}]r[1-2\epsilon(r(t))]F_g$$

for $r < r_0$ where $r_0 = \frac{2}{c^2}$. Together these equations allow for an additional correction of $-0.00358a$ onto the prediction of general relativity for the effect of Mercury perihelium rotation. Currently, the best measurement for the precession is 43" per century. General relativity predicts around

$$[42.98" + 1.289"(J_{2\odot}/10^-5)] percentury \tag{18}$$

where $J_{2\odot}$ is the solar quadropole moment[5]. The best measurements of the solar quadropole moment are $(2.5 \pm 0.2)X10^{-5}$, combined with the previous correction, the results are consistent with experimental data.

5 Conclusions

It has been shown that at least one phenomenon inconsistent with current theory is, in fact, consistent given a fractional universe; however, further tests must be done in order to verify that a fractional universe truly exists. As measurements become more accurate, their divergence from theoretical predicted values will become more apparent and therefore open up many more opportunities for the verification of fractional dimensions; however, given that all the benefits of the theory arise no matter how small the divergence from integer value becomes, the theory of fractional dimensions could possibly catch on by acting as a mathematical limit necessary for unified theory.

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