Meson Condensation and Holographic QCD

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science in Physics from The College of William and Mary

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Accepted for Honors
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April 25th, 2012

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April 13, 2012

Abstract

Quantum Chromodynamics (QCD), which describes the strong nuclear force, is difficult to solve both analytically and numerically. We use a five-dimensional model of QCD motivated by the Anti-de Sitter/Conformal Field Theory (AdS/CFT) duality originally proposed by Maldacena. We discuss how this model and other variations in the literature represent chiral symmetry breaking, and test whether these models correctly reproduce chiral symmetry beyond leading order. We compare the predictions of two models, one which is correct to leading order, and another which is correct beyond leading order. The model correct to leading order does not properly predict the pion condensation phase transition, whereas the model with the correct beyond leading order qualitatively agrees with chiral perturbation theory in its description of pion condensation. Using these two models, we calculate certain observables and find agreement with experiment to within 15%.

1 Introduction

Holographic QCD is a five-dimensional model of QCD based on the AdS/CFT correspondence conjectured by Maldacena. The AdS/CFT correspondence is a duality between an N-dimensional theory with gravity and an N-1-dimensional theory without gravity [1]. Due to the difficulty of calculating observables in QCD, simple models may be of great use as long as the fundamentals of the physics are represented within the model. Holographic QCD can be used to predict masses, decay constants, and hadron couplings at an accuracy typically around the 10-15% level.

Two of the defining characteristics of QCD are confinement and asymptotic freedom. Confinement is the statement that no states carry non-zero color charge, which is carried by quarks and gluons; therefore there cannot exist a state of an unbound quark or gluon. This may be understood heuristically as the statement that forces between quarks do not decrease with distance, therefore once two quarks are separated by a sufficient amount, it is energetically favorable to create a quark-antiquark pair rather than moving the initial quarks further apart. Asymptotic freedom in QCD is the feature that the coupling between two quarks goes to zero at high energies or small distances. Confinement is the main motivation for these models, because it prevents the use of perturbation theory. Using AdS/CFT, it is possible to model the theory in a perturbative way, thus allowing much easier calculation.

The model proposed in Ref. [2] modeled chiral symmetry, an approximate symmetry that is broken in QCD, and included a symmetry breaking term. However, it was found in Ref. [3] that the boundary conditions in this model were inconsistent with chiral symmetry breaking beyond the leading order term. It was also found that parameterizing the expansion of a scalar field in the model differently with the same boundary conditions would result in the correct representation of chiral symmetry breaking. The initial goal of this project was to investigate the consequences of the field redefinition. This was an extension of the work done by Ron Wilcox, a member of the class of 2011 whose honors project focused on the same model. Wilcox considered the Gell-Mann - Oakes - Renner (GOR) relation:

$$m_\pi^2 f_\pi^2 = (m_u + m_d)\langle q\bar{q}\rangle = 2m_q\sigma, \tag{1}$$

where m_π is the mass of the pion, f_π is the pion decay constant, and m_u, m_d are the masses of the up and down quarks, respectively. The parameter m_q is identified as the quark mass; the model assumes that $m_q = m_u = m_d$. The parameter $\sigma = \langle q\bar{q}\rangle$ is the chiral condensate. This relation should hold when σ is complex, according to chiral perturbation theory (χPT), but it did not in the model. The result Wilcox found was:

$$m_\pi^2 f_\pi^2 \ln\left(\frac{\sigma}{\sigma*}\right) = 2m_q(\sigma - \sigma*), \tag{2}$$

which reduces to eq. (1) when σ is real [4]. The chiral condensate is real, but in χPT, it is simply a parameter which is not *a priori* restricted to be real.

From the model, it is possible to predict the details of the pion condensation phase transition. Using the isospin chemical potential μ_I, the model predicts that when $m_\pi < |\mu_I|$, a pion condensate forms [3]. However, the model does not correctly identify the order of the phase transition. χPT predicts a second-order transition whereas the model predicts a first-order transition. Once the representation of chiral symmetry breaking was corrected, it was shown that the pion phase transition predicted by the model was qualitatively the same as that predicted by χPT [3].

While investigating the consequence of the field redefinition, it became relevant how similar holographic models included chiral symmetry breaking. Models with different gauge choices also claimed to have the correct representation of chiral symmetry breaking. The purpose of this project became to investigate similar models and see what could be learned from how others treated chiral symmetry. In particular, we focused on the models presented in Refs. [5] and [6].

2 Holographic QCD

This discussion will cover 4 different models. The first two are the "old" and "new" models, i.e. the initial model discussed in Ref. [2] as the "old" and the model with the corrected chiral symmetry breaking from Ref. [3] as the "new" model. The latter two models are those discussed by Da Rold and Pomerol in Ref. [5] and by Domenech, Panico, and Wulzer in Ref. [6]. Since these models share certain characteristics, we will first discuss the commonalities between the models, then discuss each individually.

2.1 Characteristics of the models

Holographic QCD is a five-dimensional model. All fields are functions of the four-vector x and a coordinate z. The four-vector x is defined such that x_0 is the time coordinate and $x_{1,2,3} = \vec{x}_{1,2,3}$. The coordinate z is restricted $0 < z \leq z_m$, where z_m is a parameter of the confining scale for QCD, referred to as the infrared (IR) boundary. The exact value of z_m will be fixed in each model. The lower bound for z is the ultraviolet (UV) boundary ϵ, which is an arbitrarily small number generally considered in the limit $\epsilon \to 0$. We use Einstein summation convention, so that common upper and lower indices are summed over:

$$A^N B_N = \sum_N A^N B_N, \tag{3}$$

All Greek indices are indices over $0, 1, 2, 3$. Indices represented by capital Latin letters cover $0, 1, 2, 3, z$. Indices are lowered with the metric g_{MN} and raised by its inverse, g^{MN}:

$$A_N = g_{MN} A^M; A^N = g^{MN} A_M, \tag{4}$$

The metric g_{MN} in the models is:

$$g_{MN} = \frac{R^2}{z^2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{5}$$

where R is the AdS curvature. The models we will be dealing with include fields X, a scalar field, and A_R^M and A_L^M, vector fields coupling to currents of right- and left-handed quarks, respectively. It is sometimes convenient to define vector fields A^M and V^M, respectively the axial vector and vector fields, as follows:

$$A^M = \frac{A_L^M - A_R^M}{2}, \tag{6}$$

$$V^M = \frac{A_L^M + A_R^M}{2}, \tag{7}$$

The models considered herein are hard-wall models, meaning that the boundaries in the fifth dimension are at fixed values of the coordinate z. As a result, the masses of heavy particles predicted by the models are proportional to n^2, for some integer n. An analogy to this is the problem of the particle-in-a-box of quantum mechanics. However, in QCD, rest frame masses grow as radial quantum number n, not as n^2. Thus hard-wall models are effective for low energy particles, but are not accurate at higher energies. As a result, the models consider low-energy bosons.

2.2 The "old" model

The action of the model can be written as follows:

$$S = \int d^5x \sqrt{|g|} \mathrm{Tr} \left\{ |DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2) \right\}, \tag{8}$$

where g is the determinant of g_{MN}, D_M is the covariant derivative $D_M X = \partial_M X - iA_{LM}X + iXA_{RM}$, $F_{L,R}^2 = (F_{L,R})_{MN}(F_{L,R})^{MN}$, $|DX|^2 = (D_M X^\dagger)(D^M X)$, and $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$. There is a degree of gauge freedom, so A_{Lz}, A_{Rz} are both set to 0. The parameter g_5^2 is found to be $\frac{12\pi^2}{N_c}$, where $N_c = 3$ is the number of colors in QCD. The AdS curvature R is set to 1 for simplicity [2].

Substituting A and V according to eqs. (6) and (7) for A_L and A_R in eq. (8) yields the following, to second order in the fields:

$$S = \int d^5x\sqrt{|g|}\mathrm{Tr}\left\{|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_A^2 + F_V^2)\right\}, \tag{9}$$

where now $D_\mu X = \partial_\mu X + i[V_\mu T, X] - i\{A_\mu T, X\}$. The matrices $T^a = \frac{\sigma^a}{2}$, where the σ^a are the Pauli sigma matrices.

The background solution for X is $\langle X\rangle = \frac{1}{2}M_q z + \frac{1}{2}\Sigma z^3$. The matrices Σ, M_q are assumed to be $m_q\mathbf{1}, \sigma\mathbf{1}$. Thus X is a scalar times the identity matrix. The expression for $D_M X$ may be simplified for calculations about the background solution because the commutator must cancel.

From eq. (9), we can derive the equations of motion for the axial vector and vector fields. This yields:

$$\partial_z\left(\frac{1}{z}\partial_z V_\mu(q,z)\right) + \frac{q^2}{z}V_\mu(q,z) = 0, \tag{10}$$

where $V(q,z)$ is the Fourier transform of $V(x,z)$:

$$V(q,z) = \int d^4x V(x,z)e^{iq\cdot x}, \tag{11}$$

Likewise for $A(q,z)$, the Fourier transform of $A(x,z)$, we get the following:

$$\partial_z\left(\frac{1}{z}\partial_z A_\mu(q,z)\right) + \frac{q^2}{z}A_\mu(q,z) - \frac{g_5^2 v(z)^2}{z^3}A_\mu = 0, \tag{12}$$

where $v(z) = m_q z + \sigma z^3$, which is twice the background solution to X. These fields are normalized such that:

$$\int_\epsilon^{z_m}\frac{dz}{z}\psi(z)^2 = 1, \tag{13}$$

where ψ is the solution to the equations of motion in eqs. (10) or (12) with boundary conditions to be specified later.

2.3 The "New" Model

The action and boundary conditions are the same for the new model:

$$S = \int d^5x\sqrt{|g|}\mathrm{Tr}\left\{|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\right\}, \tag{14}$$

We also keep the gauge choice $A_{Rz} = A_{Lz} = 0$. The physics is changed in this model by the parameterization of the expansion of the field X. In the old model, X was expanded about the background solution in the following way:

$$X(x,z) = \left(\frac{1}{2}(\tilde{m}_q z + \sigma z^3) + \tilde{S}(x,z)\right)\tilde{U}(x,z), \tag{15}$$

The new model description is, taking $\tilde{m}_q$ to $-2m_q$:

$$X(x,z) = -m_q z + \left(\frac{\sigma}{2}z^3 + S(x,z)\right)U(x,z), \tag{16}$$

where $S(x,z)$ is a hermitian matrix not to be confused with the action S, the integral of $\mathcal{L}$ over all space, and $U(x,z)$ is a unitary matrix defined as $U = e^{i\pi^a(x,z)\sigma^a}$, where π is the pion field, the σ^a are the Pauli sigma matrices, and $a \in \{1,2,3\}$ is a gauge index. $\tilde{S}(x,z)$ and $\tilde{U}(x,z)$ are hermitian and unitary matrices, respectively, the tildes are used to signify that these two matrices are not necessarily the same as $S(x,z)$ and $U(x,z)$. It is possible to parameterize any matrix as the product of a hermitian matrix and a unitary matrix. Therefore with the same boundary conditions placed on X, the field would be no different. The difference between the models is that the boundary conditions on $S(x,z)$ and $\pi(x,z)$ differ. The boundary conditions for S and π are [3]:

$$\pi(x,\epsilon) = S(x,\epsilon) = S(x,z_m) = 0, \tag{17}$$

$$\partial_z \pi(x,z)|_{z_m} = \frac{\tilde{m}_q}{\sigma z_m^3}\pi \tag{18}$$

If these boundary conditions are applied equally to eqs. (15) and (16), then there are different boundary conditions on X due to the different parameterization. The boundary conditions are modified by changing $\tilde{m}_q \to -2m_q$, which when decomposed into the Kaluza-Klein mode $\pi(x,z) = \pi(x)\psi(z)$, sets the IR boundary condition:

$$\partial_z \psi(z)|_{z_m} = -\frac{2m_q}{\sigma z_m^3}\psi(z_m) \tag{19}$$

Since the action is the same, the equation of motion for $V(q,z)$ is unaltered. This does result in $v(z)$ changing from $v(z) = \tilde{m}_q z + \sigma z^3$ to $v(z) = -2m_q z + \sigma z^3$. Because of this, the equation of motion for $A(q,z)$ changes.

2.4 The Model of Da Rold and Pomerol

The action in the model of Da Rold and Pomerol is the same as the action in the previous models up to normalizations of the fields:

$$S = M_5 \int d^5x \sqrt{|g|} \mathrm{Tr} \left[-\frac{1}{4} F_L^2 - \frac{1}{4} F_R^2 + \frac{1}{2} |D_M X|^2 - \frac{1}{2} M_\Phi^2 |X|^2 \right], \tag{20}$$

where M_5 is a gauge coupling, and $M_\Phi^2 = \frac{3}{R^2}$ is a parameter of the model, where R is the AdS curvature, and as before $D_M X = \partial_M X + iA_{LM} X - iXA_{RM}$, and A_L, A_R are the vector fields coupled to currents of left- and right-handed quarks. The definitions for A and V used by Da Rold and Pomerol differ by a factor of $\sqrt{2}$ from the previously established definitions, but that has been accounted for in the following equations. Da Rold and Pomerol make a more complicated gauge choice, by adding the gauge fixing term to the Lagrangian [5]:

$$\mathcal{L}_{gf}^V = -\frac{M_5 a(z)}{\xi_V} \mathrm{Tr} \left[\partial_\mu V_\mu - \frac{\xi_V}{a(z)} \partial_z (a(z) V_z) \right]^2, \tag{21}$$

$$\mathcal{L}_{gf}^A = -\frac{M_5 a(z)}{\xi_A} \mathrm{Tr} \left[\partial_\mu A_\mu - \frac{\xi_A}{a(z)} \partial_z (a(z) A_z) - \xi_A a^2(z) v(z) \pi(x,z) \right]^2, \tag{22}$$

where $\xi_{A,V}$ are arbitrary parameters used to set the gauge and $\pi(x,z)$ is the pion field. The gauge is set by taking $\xi_{A,V} \to \infty$, thus forcing the terms of order ξ to go to zero. Therefore the gauge is:

$$\partial_z (a(z) V_z) = 0, \tag{23}$$

$$\pi(x,z) = -\frac{1}{a^3(z) v(z)} \partial_z (a(z) A_z), \tag{24}$$

Plugging these equations into the gauge fixing terms in eqs. (21) and (22) cancels the terms of zeroth and first order in ξ in the gauge fixing term, leaving a term proportional to ξ^{-1}, which does not contribute in the limit $\xi_{A,V} \to \infty$. Thus these gauge fixing terms may be added to the Lagrangian to set the gauge, without changing the physics within the original Lagrangian.

2.5 The Model of Domenech, Panico and Wulzer

The action for the scalar field X is as follows [6]:

$$S_X = M_5 \int d^5x a^3(z) \left[\mathrm{Tr}|D_M X|^2 - a^2(z) M_{bulk}^2 \mathrm{Tr}|X|^2 \right], \tag{25}$$

where M_{bulk}^2 is a parameter of the model and M_5 is the gauge coupling. In this model [6], they choose the same gauge as in the previous section, with a gauge fixing term very similar to that in the previous section:

$$\mathcal{L}_{gf}^V = -\frac{2M_5 a(z)}{\xi_V} \mathrm{Tr} \left[\partial_\mu V^\mu - \frac{\xi_V}{a(z)} \partial_z (a(z) V_z) \right]^2, \tag{26}$$

$$\mathcal{L}_{gf}^A = -\frac{2M_5 a(z)}{\xi_A} \mathrm{Tr} \left[\partial_\mu A^\mu - \frac{\xi_A}{a(z)} \partial_z (a(z) A_z) - \xi_A a^2(z) v(z) \pi(x,z) \right]^2, \tag{27}$$

where once again $\xi_{V,A}$ are arbitrary parameters used to fix the gauge, $v(z)$ is the vacuum expectation value of the scalar field, and $\pi(x,z)$ is the pion field. Again taking $\xi_{A,V} \to \infty$ sets the gauge:

$$\partial_z (a(z) V_z) = 0, \tag{28}$$

$$\pi(x,z) = -\frac{1}{a^3(z) v(z)} \partial_z (a(z) A_z), \tag{29}$$

3 Chiral Symmetry Breaking in the Models

Chiral symmetry is a symmetry in which quark fields charged under $SU(2)_L$, i.e. left-handed fields, transform as:

$$q_L \to U_L q_L, \tag{30}$$

where U_L is a unitary matrix. Those fields charged under $SU(2)_R$, i.e. right-handed fields, transform as:

$$q_R \to q_R U_R^\dagger, \tag{31}$$

where U_R is a unitary matrix. The scalar field X transforms as follows:

$$X \to U_L X U_R^\dagger, \tag{32}$$

This implies that the unitary matrix U with which we expand about the vacuum expectation value of X transforms in the same way. The chiral symmetry is explicitly broken in the chiral Lagrangian by a term proportional to:

$$\mathrm{Tr}[UM_q^\dagger + M_q U^\dagger], \tag{33}$$

The term inside the trace transforms like:

$$UM_q^\dagger + M_q U^\dagger \to U_L U U_R^\dagger M_q^\dagger + M_q U_R U^\dagger U_L^\dagger \neq U_L(UM_q^\dagger + M_q U^\dagger)U_R^\dagger, \tag{34}$$

Since this term is not invariant under chiral symmetry, except for the case $U_L = U_R$, the chiral symmetry is broken. The matrix $U = e^{i\pi^a \sigma^a}$ contains terms of all orders in the pion field π, therefore chiral symmetry is broken beyond the leading order in the pion field. Since M_q multiplies this term, we say that a non-zero quark mass explicitly breaks the chiral symmetry.

3.1 The Pion and Goldstone's Theorem

Goldstone's theorem states, in its relativistic form, that when there is an exact symmetry which is spontaneously broken, then there is a corresponding massless particle. When there is an approximate symmetry that is broken, there is a low-mass particle corresponding to the symmetry. This particle is the Goldstone boson corresponding to the field. Since chiral symmetry is an approximate symmetry, when it is broken, there is a low-mass particle. For chiral symmetry, this particle is the pion.

3.2 Chiral Symmetry Breaking in the "Old" Model

Taking the scalar field action from eq. (9) and substituting in the expansion for X from eq. (15), perturbing only U about $\langle X \rangle$ with $S(x,z) = 0$, which is possible because to second order S decouples from the other fields, we get:

$$S \supset \int d^5x \sqrt{|g|}\mathrm{Tr}\left\{ \left| D\left[\frac{1}{2}(m_q z + \sigma z^3)U(x,z)\right]\right|^2 + 3\left|\frac{1}{2}(m_q z + \sigma z^3)U(x,z)\right|^2 \right\}, \tag{35}$$

Since U is unitary, the $|X|^2$ term looks like the following:

$$|X|^2 = \frac{1}{2}(m_q z + \sigma z^3)U(x,z)U^\dagger(x,z)\frac{1}{2}(m_q z + \sigma z^3)^\dagger = \frac{1}{4}|m_q z + \sigma z^3|^2, \tag{36}$$

Therefore this cannot contribute to chiral symmetry breaking because it lacks any factors of $\pi(x,z)$. Let us now consider the $|DX|^2$ term:

$$|DX|^2 = D_M \left[\frac{1}{2}(m_q z + \sigma z^3)U(x,z)\right] \left(D^M \left[\frac{1}{2}(m_q z + \sigma z^3)U(x,z)\right]\right)^\dagger, \tag{37}$$

If we substitute the Kaluza-Klein mode of the pion field $\pi(x,z) = \pi(x)\psi(z)$ and turn off the coupling g_5^2, effectively driving the fields A, V to zero so that we may replace $D_M X$ with $\partial_M X$, then we get the following:

$$\begin{aligned} |DX|^2 &= \frac{1}{4}|m_q z + \sigma z^3|^2 \partial_\mu [U(x,z)] \partial^\mu [U^\dagger(x,z)] \\ &\quad -\frac{1}{4}\partial_z[(m_q z + \sigma z^3)U(x,z)]\frac{1}{z^2}\partial_z[U^\dagger(x,z)(m_q z + \sigma z^3)^\dagger], \end{aligned} \tag{38}$$

where ∂^z has the index lowered with $g_{zz} = -\frac{1}{z^2}$. Next we take the derivatives:

$$\partial_\mu U(x,z) = i\pi^{a\prime}(x)\psi(z)\sigma^a U(x,z), \tag{39}$$

$$\partial_z U(x,z) = i\pi^a(x)\psi'(z)\sigma^a U(x,z), \tag{40}$$

Since the σ matrices have the property $\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^a_b$, we have $\mathrm{Tr}(\pi^a\sigma^a\sigma^b\pi^b) =\mathrm{Tr}(\pi^a\pi^a)$, summing over $a, b \in \{1,2,3\}$. Thus, considering only the terms with a nonzero trace:

$$\begin{aligned} |DX|^2 &= \frac{1}{4}|(m_q z + \sigma z^3)\psi(z)|^2 \partial_\mu \pi^a(x) \partial^\mu \pi^a(x) \\ &\quad -\frac{1}{4z^2}\left[(m_q + 3\sigma z^2)U(x,z) + i(m_q z + \sigma z^3)\pi^a(x)\psi'(z)\sigma^a U(x,z)\right] \\ &\quad \cdot \left[U^\dagger(x,z)(m_q + 3\sigma z^2)^\dagger - iU^\dagger(x,z)\sigma^a\psi'(z)\pi^a(x)(m_q z + \sigma z^3)^\dagger\right], \end{aligned} \tag{41}$$

Assuming m_q, σ real, this is:

$$|DX|^2 = \frac{1}{4}\left[v^2(z)\psi^2(z)\partial_\mu\pi^a(x)\partial^\mu\pi^a(x) - \frac{1}{z^2}\left[\left(m_q + 3\sigma z^2\right)^2 + v^2(z)(\pi^a(x))^2\psi'^2(z)\right]\right], \tag{42}$$

The term $v^2(z)(\pi^a(x))^2\psi'^2(z)$ breaks chiral symmetry, but it is only to second order in the pion field. For chiral symmetry to be correctly represented, we must have the correct pattern of chiral symmetry breaking beyond the leading order. Thus we must either change the boundary conditions, e.g. choose a non-zero $S(x,z)$, or change the expansion of X about the vacuum expectation value

[3].

3.3 Chiral Symmetry Breaking in the "New" Model

Changing the boundary conditions in the old model results in complicated non-linear boundary conditions [3]. Thus it is easier to re-parameterize the field X as in eq. (16) with boundary conditions given in eqs. (17), (18), and (19). The action is the same as in eq. (9), and as before we will ignore the terms that do not include the scalar field, so we are left with a term proportional to $|X|^2$ and a term proportional to $|DX|^2$. Let us first consider the $|X|^2$ term, once again keeping $S(x,z) = 0$:

$$\begin{aligned} |X|^2 &= \left[m_q z + \frac{\sigma}{2} z^3 U(x,z)\right]\left[m_q^\dagger z + \frac{U^\dagger(x,z)\sigma^\dagger}{2} z^3\right] \\ &= m_q^2 z^2 + \frac{\sigma^2}{4} z^6 + \frac{z^4}{2}\left[m_q U^\dagger(x,z)\sigma^\dagger + \sigma U(x,z) m_q^\dagger\right], \end{aligned} \tag{43}$$

Since σ is real and proportional to $\mathbf{1}$, we can rewrite the last term:

$$|X|^2 \supset \frac{\sigma z^4}{2}[m_q U^\dagger + U m_q^\dagger], \tag{44}$$

To ensure that the chiral symmetry breaking term does not cancel with another term within the Lagrangian, we will now look at the $|DX|^2$ term, keeping the assumptions from the previous section in place so that $D_M = \partial_M$

$$\begin{aligned} |DX|^2 &= \partial_\mu \left[m_q z + \frac{\sigma}{2} z^3 U(x,z)\right]\left[\partial^\mu \left(m_q z + \frac{\sigma}{2} z^3 U(x,z)\right)\right]^\dagger \\ &\quad -\frac{1}{z^2}\partial_z \left[m_q z + \frac{\sigma}{2} z^3 U(x,z)\right]\left[\partial_z \left(m_q z + \frac{\sigma}{2} z^3 U(x,z)\right)\right]^\dagger, \end{aligned} \tag{45}$$

where once again ∂^z has the index lowered with g_{zz}. Using eqs. (39) and (40) we obtain:

$$\begin{aligned} |DX|^2 &= \frac{|\sigma|^2}{4} z^6 [i\partial_\mu \pi^a(x)\psi(z)\sigma^a U(x,z)][-iU^\dagger(x,z)\sigma^b\psi(z)\partial^\mu\pi^b(x)] \\ &\quad -\frac{1}{z^2}\left[m_q + \frac{3\sigma}{2} z^2 U(x,z) + \frac{i\sigma}{2} z^3 \pi^a(x)\psi'(z)\sigma^a U(x,z)\right] \\ &\quad \cdot \left[m_q^\dagger + U^\dagger(x,z)\frac{3\sigma^\dagger}{2} z^2 - iU^\dagger(x,z)\sigma^b\psi'(z)\pi^b(x)\frac{\sigma^\dagger}{2}\right], \end{aligned} \tag{46}$$

Expanding and simplifying with $\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^a_b$, we have:

$$
\begin{aligned}
|DX|^2 \quad = \quad & \frac{|\sigma|^2\psi^2(z)}{4} z^6 \partial_\mu \pi^a(x) \partial^\mu \pi^a(x) \\
& - \frac{|m_q|^2}{z^2} - \frac{9|\sigma|^2}{4} z^2 - \frac{|\sigma|^2 \psi'^2(z)}{4} \pi^a(x)\pi^a(x) \\
& - \frac{m_q \sigma^\dagger}{2z^2} \left(U^\dagger(x,z) 3z^2 - iU^\dagger(x,z)\sigma^b \psi'(z) \pi^b(x) \right) \\
& - \left(3z^2 U(x,z) + iz^3 \pi^a(x) \psi'(z) \sigma^a U(x,z) \right) \frac{m_q^\dagger \sigma}{2z^2}, \qquad (47)
\end{aligned}
$$

Any additional terms cancel as long as $\pi(x), \psi(z)$ are real. We may neglect the terms with σ^a remaining since the trace of the sigma matrices is zero, and we will be taking the trace in the Lagrangian. Choosing σ real and proportional to $\mathbf{1}$, we are left with:

$$
\begin{aligned}
|DX|^2 \quad = \quad & \frac{|\sigma|^2\psi^2(z)}{4} z^6 \partial_\mu \pi^a(x) \partial^\mu \pi^a(x) - \frac{|m_q|^2}{z^2} - \frac{9|\sigma|^2}{4} z^2 - \frac{|\sigma|^2 \psi'^2(z)}{4} \pi^a(x)\pi^a(x) \\
& - \frac{3\sigma}{2} \left[m_q U^\dagger(x,z) + U(x,z) m_q^\dagger \right], \qquad (48)
\end{aligned}
$$

Plugging the terms that contain the pattern of chiral symmetry breaking into the action, we get:

$$
\begin{aligned}
S \quad \supset \quad & \int d^5x \sqrt{|g|} \mathrm{Tr} \left[|DX|^2 + 3|X|^2 \right] \\
\supset \quad & \int d^5x \sqrt{|g|} \mathrm{Tr} \left[\left(\frac{3\sigma z^4}{2} - \frac{3\sigma}{2} \right) \left[m_q U^\dagger + U m_q^\dagger \right] \right], \qquad (49)
\end{aligned}
$$

Therefore there is a chiral symmetry breaking term in the new model.

3.4 Chiral Symmetry Breaking in Domenech et. al.

Domenech et al. show in Ref. [6] that their model has the proper representation of chiral symmetry breaking. Here, we show the same process in additional detail. The field X is parameterized in a similar way as in the "old" model, since U multiplies all of $v(z)$:

$$
X = (v\mathbf{1} + S) e^{i\pi(x,z)/v} = (v\mathbf{1} + S) U, \qquad (50)
$$

where $v(z) \equiv \langle X \rangle$ is:

$$v(z) = \frac{z_m^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \left(\frac{z}{z_m}\right)^{\Delta^+} (\xi - M_q)\mathbf{1} + \frac{1}{z_m^{2\alpha} - \epsilon^{2\alpha}} \left(\frac{z}{z_m}\right)^{\Delta^-} (z_m^{2\alpha} M_q - \epsilon^{2\alpha}\xi)\mathbf{1}, \tag{51}$$

where $\Delta^{\pm} = 2 \pm \alpha$, and α is a parameter of the model. For the old and new models, $\alpha = 1$, in which case there would be a z and a z^3 term, as is in $v(z)$ in those models. The boundary conditions on $\pi(x, z)$ and $S(x, z)$ are:

$$S(x, z_m) = S(x, \epsilon) = \pi(x, z_m) = \pi(x, \epsilon) = 0 \tag{52}$$

X is expanded in momentum terms:

$$X = X^{(0)}(z) + X^{(2)}(p, z, \hat{A}, \hat{V}, U, M_q) + X^{(4)}(p, z, \hat{A}, \hat{V}, U, M_q), \tag{53}$$

where $\hat{A}, \hat{V}$ are A, V evaluated at the UV boundary z_m, and the superscript indicates the order of the momentum term. Terms with odd powers of momentum vanish [6]. After integration by parts, the action contains the term:

$$S \supset M_5 \int_{UV,IR} d^4x a^3(z) \mathrm{Tr}[X^\dagger \partial_z X + (\partial_z X)^\dagger X], \tag{54}$$

Domenech et al. place the following IR boundary conditions on the $X^{(i)}$:

$$\begin{aligned} &X^{(0)}(x, z = z_m) = \xi\mathbf{1}, \\ &X^{(i)}(x, z = z_m) = 0, \ i \geq 2, \end{aligned} \tag{55}$$

where ξ is a parameter of the model. The UV boundary conditions are:

$$\begin{aligned} &X^{(0)}(x, z = \epsilon) = 0, \\ &X^{(2)}(x, z = \epsilon) = U \left(\frac{\epsilon}{z_m}\right)^{\Delta^-} M_q^\dagger, \\ &X^{(i)}(x, z = \epsilon) = 0, \ i \geq 4, \end{aligned} \tag{56}$$

Since M_q is of second order in momentum, the only terms that contribute to the chiral symmetry breaking term are of second order in momentum:

$$S \supset M_5 \int_{UV,IR} d^4 x a^3(z) \mathrm{Tr}[X^{(0)\dagger}\partial_z X^{(2)} + X^{(2)\dagger}\partial_z X^{(0)} + \text{ h.c.}], \tag{57}$$

where h.c. is the hermitian conjugate of the previous terms. On the IR boundary, $X^{(2)}$ goes to zero, thus the term $X^{(2)\dagger}\partial_z X^{(0)}$ vanishes. Although $X^{(2)}$ is zero, the boundary conditions do not place restrictions on the z-derivative of $X^{(2)}$, therefore the term $X^{(0)\dagger}\partial_z X^{(2)}$ does not necessarily vanish on the IR boundary; however, Domenech et al. consider the UV boundary only. On the UV boundary, $X^{(0)}$ is zero, therefore the only term in the UV that contributes is:

$$S \supset M_5 \int_{UV} d^4 x a^3(z) \mathrm{Tr}[X^{(2)\dagger}\partial_z X^{(0)} + \text{ h.c.}], \tag{58}$$

The definition of $X^{(0)}$ is X in eq. (51) in the limit $M_q \to 0$ [6]:

$$X^{(0)}(z) = \frac{z_m^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \left(\frac{z}{z_m}\right)^{\Delta^+} \xi \mathbf{1} - \frac{\epsilon^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \left(\frac{z}{z_m}\right)^{\Delta^-} \xi \mathbf{1}, \tag{59}$$

Taking the z-derivative:

$$\partial_z X^{(0)}(z) = \Delta^+ \frac{z_m^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \frac{z^{\Delta^+ - 1}}{z_m^{\Delta^+}} \xi \mathbf{1} - \Delta^- \frac{\epsilon^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \frac{z^{\Delta^- - 1}}{z_m^{\Delta^-}} \xi \mathbf{1}, \tag{60}$$

Evaluated at the UV boundary:

$$\partial_z X^{(0)}(\epsilon) = \Delta^+ \frac{z_m^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \frac{\epsilon^{\Delta^+ - 1}}{z_m^{\Delta^+}} \xi \mathbf{1} - \Delta^- \frac{\epsilon^{2\alpha}}{z_m^{2\alpha} - \epsilon^{2\alpha}} \frac{\epsilon^{\Delta^- - 1}}{z_m^{\Delta^-}} \xi \mathbf{1}, \tag{61}$$

Since $\Delta^+ = \Delta^- + 2\alpha$, this simplifies further:

$$\partial_z X^{(0)}(\epsilon) = 2\alpha \frac{\epsilon^{\Delta^- + 2\alpha - 1}}{(z_m^{2\alpha} - \epsilon^{2\alpha}) z_m^{\Delta^-}} \xi \mathbf{1}, \tag{62}$$

Substituting back into eq. (58) for $\partial_z X^{(0)}$ and $X^{(2)}$, and plugging in $z = \epsilon$, we get:

$$S \supset 2\alpha \xi M_5 \int d^4 x \left(\frac{L}{\epsilon}\right)^3 \mathrm{Tr}\left[M_q U^\dagger \frac{\epsilon^{2\Delta^- + 2\alpha - 1}}{(z_m^{2\alpha} - \epsilon^{2\alpha}) z_m^{2\Delta^-}} + \text{ h.c.}\right], \tag{63}$$

Substituting in for $\Delta^- = 2 - \alpha$, we get:

$$S \supset 2\alpha \xi L^3 M_5 \int d^4 x \mathrm{Tr}\left[M_q U^\dagger \frac{1}{z_m^4 - \epsilon^{2\alpha} z_m^{2\Delta^-}} + \text{ h.c.}\right], \tag{64}$$

Finally, setting the AdS curvature $R = z_m$, as was done in this model, and taking the limit $\epsilon \to 0$, we get:

$$S \supset \frac{2\alpha\xi}{R} M_5 \int d^4x \text{Tr}[M_q U^\dagger + \text{ h.c.}], \tag{65}$$

Substituting in the hermitian conjugate of $M_q U^\dagger$:

$$S \supset \frac{2\alpha\xi}{R} M_5 \int d^4x \text{Tr}[M_q U^\dagger + U M_q^\dagger], \tag{66}$$

Therefore we can see that we have the chiral symmetry breaking term in the model.

3.5 Chiral Symmetry Breaking in Da Rold and Pomerol

Da Rold and Pomerol use the same gauge choice as Domenech et al. However, their definition of $v(z) \equiv \langle X \rangle$ and boundary conditions on X differ. $v(z)$ is:

$$v(z) = \frac{\tilde{M}_q z_m^3 - \xi\epsilon^2}{R z_m (z_m^2 - \epsilon^2)} z + \frac{\xi - \tilde{M}_q z_m}{R z_m (z_m^2 - \epsilon^2)} z^3 \tag{67}$$

where $\tilde{M}_q$ and ξ are defined as follows:

$$\tilde{M}_q = \frac{R}{\epsilon} v(\epsilon) \tag{68}$$

$$\xi = R v(z_m) \tag{69}$$

The expansion about X is the same as in eq. (50), with $v(z)$ defined in the above equation. The IR boundary conditions for S and A_z, which has been related to the pion field by the gauge choice in eq. (22), are chosen:

$$[M_5 \partial_z + 2a(z) m_S^2] S|_{z_m} = 0 \tag{70}$$

$$A_z(x, z_m) = 0 \tag{71}$$

where m_S is a mass term associated with the field S. Da Rold and Pomerol then claim that it is possible to derive a Lagrangian to $\mathcal{O}(p^2)$:

$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \text{Tr}[D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U] \tag{72}$$

where F_π is the pion decay constant, U is defined differently from before as $U = e^{i\sqrt{2}\pi/F_\pi}$ and $\chi = 2B_0(M_q + s + ip_s)$ where B_0 is a constant, and s and p_s are fictitious scalar and pseudoscalar sources used to calculate correlation functions. These terms include the chiral symmetry breaking term since this newly defined U still contains the pion field to an arbitrarily high order, and it is multiplied by $M_q^\dagger$, with the hermitian conjugate added:

$$\mathcal{L}_2 = \frac{F_\pi^2}{4}\mathrm{Tr}[U^\dagger M_q + M_q U] \tag{73}$$

Therefore if this term is correct, the model includes chiral symmetry breaking beyond leading order. There were initial doubts as to whether this model contained chiral symmetry breaking terms beyond leading order or whether, like Ref. [2], it was only assumed. Then it was found in Ref. [6] that it was possible to have the correct representation of chiral symmetry breaking with the expansion of the field X written as in eq. (50).

4 Model Results and Pion Physics

4.1 Calculating Observables in the "Old" and "New" Models

The boundary conditions for the Kaluza-Klein modes of $V(q, z)$ are:

$$\begin{aligned} V(m_\rho, \epsilon) &= 0, \\ \partial_z V(m_\rho, z_m) &= 0, \end{aligned} \tag{74}$$

Non-trivial solutions for $V(q, z)$ from eq. (10) are of the form:

$$V(q, z) = zJ_1(m_\rho z), \tag{75}$$

where J_1 is the Bessel J function. The wavefunction $V(q, z)$ describes the rho meson, therefore solving such that $V(q, z_m)$ is zero, where q is the mass of the rho meson, 775 MeV, yields $z_m = 1/(323\text{MeV})$ [2]. The rho meson decay constant F_ρ is a parameter of how likely a rho meson is to couple to a photon:

$$F_\rho^2 = \frac{1}{g_5^2}[V''(m_\rho, 0)]^2, \tag{76}$$

where primes signify derivatives with respect to z [2]. Calculating $F_\rho^{\frac{1}{2}}$ from eq. (76) yields $F_\rho^{\frac{1}{2}} = 329$ MeV.

We calculate f_π from the following:

$$f_\pi^2 = -\frac{1}{g_5^2}\frac{\partial_z A(0,z)}{z}|_{z\to\epsilon}, \tag{77}$$

where ϵ is an arbitrarily small, non-zero value. This definition of f_π and the known value for m_q and m_π are combined with the GOR relation (1) to calculate $\sigma = (327\text{ MeV})^3$. The boundary conditions of A for the calculation of the pion decay constant are: [2]

$$\begin{aligned} A(0,\epsilon) &= 1, \\ \partial_z A(0,z_m) &= 0, \end{aligned} \tag{78}$$

Solutions for the Kaluza-Klein modes of $A(q,z)$ from eq. (12) do not have an analytic solution, but a numeric solution may be calculated using the boundary conditions:

$$\begin{aligned} A(q,\epsilon) &= 0, \\ \partial_z A(q,z_m) &= 0, \end{aligned} \tag{79}$$

From the numerical solution we find that the mass of the lowest energy Kaluza-Klein mode of the axial vector field to be 1363 MeV, thus we identify this field to be the field of the a_1 particle. We can calculate the a_1 decay constant, which is a measure of the probability the a_1 will couple to a W or Z boson, as follows:

$$F_{a_1}^2 = \frac{1}{g_5^2}[A''(m_{a_1},0)]^2, \tag{80}$$

we find $F_{a_1}^{\frac{1}{2}} = 486$ MeV.

4.2 New Model

All of the preceding calculations were accomplished using the model from Ref. [2]. The original model described low-energy hadronic physics about the background solution to X. Changing the definition of $X(x,z)$ in the model amounts to change in the boundary conditions. Adjusting for the new boundary conditions changes $v(z)$. Previously, $v(z) = m_q z + \sigma z^3$. Now $v(z) = -2m_q z + \sigma z^3$. This does not affect any of the calculations for the rho meson, but it does change the numerical values for the a_1. σ is set to 333 (MeV)^3. f_π is no longer held to the exact experimental value, with these values $f_\pi = 91.7$ Following the same procedure as earlier yield the following values: $m_{a1} = 1366$ MeV, $F_{a1}^{1/2} = 491$ MeV.

4.3 Calculations

The following table gives experimental and calculated values in MeV (asterisks indicate values fixed to experimental values):

Observable	Measured	Old Model	New Model	Da Rold	Domenech
m_π	139.6±0.0004 [2]	139.6*	139.6*	135 [7]	134 [6]
m_ρ	775.8±0.5 [2]	775.8*	775.8*	770 [7]	783 [6]
m_{a_1}	1230±40 [2]	1363	1366	1230 [7]	1320[6]
f_π	92.4±0.35 [2]	92.4*	91.7*	87 [7]	89 [6]
$F_\rho^{1/2}$	345±8 [2]	329	329	343 [7]	342 [6]
$F_{a_1}^{1/2}$	433±13 [2]	486	491	444 [7]	
RMS error		10%	11%	3%	4%

The data from Ref. [7] is optimized by choosing ξ in their model. The data for F_ρ and F_{a_1} is given using a different convention in Refs. [6] and [7]. These data were rescaled in the same convention as the old and new models for comparison. Domenech et al. did not calculate F_{a_1}. The RMS error in the model is found using the following formula [2]:

$$\epsilon_{RMS} = \left[\frac{1}{n} \sum_O (\delta O/O)^2 \right]^{1/2} \tag{81}$$

where n is the number of independent observables, and O is each independent observable. This yields an error of 10% for the old model and 11% for the new model. σ is chosen in the new model to be close to the value in the old model, therefore f_π is considered to be fixed even though it is not fixed to the exact experimental value. Correcting for the pattern of chiral symmetry breaking leads to a model that is less accurate with regards to the above observables, but only slightly; we still prefer the new model because it it qualitatively correct in the representation of chiral symmetry breaking. This table is not complete. It does not contain all possible observables that could be calculated with the models, but rather the observables which are simple to calculate, and have been calculated by the author in the old and new models and the equivalent calculations performed by the authors of the other models.

5 Conclusions

The correct representation of chiral symmetry breaking is necessary to properly model pion condensation. For calculating those observables which are simple to calculate, we see that the gauge and whether chiral symmetry breaking terms exist beyond leading order only changes the agreement with experiment between the old and new models by about 1%. We also find that despite an unknown precision, the results are accurate to approximately 10% for the observables calculated.

A Lagrangian Field Equations

Suppose we are given an action:

$$S = \int d^n x \mathcal{L}, \tag{82}$$

where $\mathcal{L}$ is the Lagrangian density, often referred to as just the Lagrangian. $\mathcal{L}$ is a function of some field A and its derivatives. It is possible to obtain an equation of motion for the field A using the following:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu}, \tag{83}$$

where $\partial_\mu A$ is shorthand:

$$\partial_\mu A = \frac{\partial A}{\partial x^\mu}, \tag{84}$$

B Finding Masses from Kaluza-Klein modes

The Kaluza-Klein modes are special solutions to the equations of motion that can be decomposed as follows:

$$A(x, z) = A(x)\psi(z), \tag{85}$$

These can be used to find the mass of a particle corresponding to a field. The relativistic energy of a particle is (setting the speed of light $c = 1$):

$$E^2 = p^2 + m^2, \tag{86}$$

where p^2 is the five-dimensional momentum squared and m^2 is the field's mass squared. Since p^2 is the sum of the momentum squared in each dimension, we can rewrite it as follows:

$$E^2 = \vec{p}^2 + p_5^2 + m^2, \tag{87}$$

Since the momentum in the fifth dimension is not measurable in 4 dimensions, the extra momentum p_5^2 term contributes to an effective mass term:

$$E^2 = \vec{p}^2 + m_{eff}^2, \tag{88}$$

In the special case where the field has no mass, we see that $m_{eff} = |p_5|$. Since p_5 is dependent on the z-derivative of the Kaluza-Klein mode, if we know the Kaluza-Klein mode, we can calculate the effective mass.

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