Fully quantized description of light-atom interaction

Reminder: semiclassical description (we treated e-m field as an external interaction

$$\hat{H}_{sc} = \underset{(atom)}{\hat{H}_a} + \hat{H}_{int} = \sum_i E_i |i\rangle\langle i| - \hat{\vec{d}}\cdot\vec{E}$$

using $\langle i|\hat{\vec{d}}\cdot\vec{e}_p|j\rangle = \rho_{ij}$ where $\vec{e}_p$ is the polarization vector

$$\hat{H}_{sc} = \sum_i E_i |i\rangle\langle i| - \sum_{\substack{i,j \\ i\neq j}} \rho_{ij} |i\rangle\langle j| \cdot E$$

Quantized e-m field

Now the system consists of two quantum objects: an electron state of an atom and the state of the photons in e-m field.

$$\hat{H} = \hat{H}_a + \underline{\hat{H}_{ph}} + \hat{H}_{int} \qquad \hat{H}_{ph} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$$\hat{H}_{int} = -\hat{\vec{d}}\,\hat{\vec{E}} \qquad \hat{\vec{E}} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,\vec{e}_p\left[\underbrace{\hat{a}e^{-i\omega t}}_{\hat{a}(t)} + \underbrace{\hat{a}^+ e^{i\omega t}}_{\hat{a}^+(t)}\right]$$

$$\hat{H}_{int} = -\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,(\hat{\vec{d}}\cdot\vec{e}_p)\,(\hat{a} + \hat{a}^+)$$

The quantum state basis: $|i,n\rangle = |i\rangle|n\rangle$

an atom is in state $|i\rangle$

$|n\rangle$ photons in the e-m field

In general $|\psi_{ph}\rangle = \sum_i \sum_n c_{i,n} |i,n\rangle$

Let's now consider the interaction of a quantized e-m field with a two-level system ($|a\rangle, |b\rangle$)

If we assume that we start at the atomic state $|a\rangle$ having $|n\rangle$ photons

$$\hat{H}_{int} |a,n\rangle = -\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, \wp_{ab} (\hat{a} + \hat{a}^+) \cancel{|a\rangle}|n\rangle\, |b\rangle\langle a| \cdot |a\rangle$$

$$= \left[-\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, \wp_{ab} \right] |b\rangle \Big(\underbrace{\sqrt{n}\,|n-1\rangle}_{\text{photon absorption}} + \underbrace{\sqrt{n+1}\,|n+1\rangle}_{\text{photon emission}} \Big)$$

Thus, as time goes on, our system can find itself in three possible states

$$|\psi(t)\rangle = C_a(t)\, |a\rangle|n\rangle\, e^{-iE_a t/\hbar}\, e^{-in\omega t} +$$
$$+ C_b(t)\, |b\rangle|n-1\rangle\, e^{-iE_b t/\hbar}\, e^{-i(n-1)\omega t} +$$
$$+ C_b'(t)\, |b\rangle|n+1\rangle\, e^{-iE_b t/\hbar}\, e^{-i(n+1)\omega t}$$

(it is convenient to keep both absorption and emission here since we don't know if $E_a > E_b$ or $E_a < E_b$)

Perturbation calculations $\qquad \omega_{ba} = \frac{E_b - E_a}{\hbar}$

$$C_b^{(1)}(t) = -\frac{i}{\hbar} \int_0^t \langle b, n-1 | \hat{H}_{int} | a, n\rangle\, e^{+i(\omega_{ba} - \omega)t'}\, dt'$$

$$C_b'^{(1)}(t) = -\frac{i}{\hbar} \int_0^t \langle b, n+1 | \hat{H}_{int} | a, n\rangle\, e^{+i(\omega_{ba} + \omega)t'}\, dt'$$

$$\langle b, n\pm 1 | \hat{H}_{int} | a, n\rangle = -\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, \wp_{ba} \begin{cases} \sqrt{n+1} & \text{"+"} \\ \sqrt{n} & \text{"-"} \end{cases}$$

$$g_{ba} = +\left[+\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \right] \frac{\wp_{ba}}{\hbar} = \frac{\mathcal{E}_0 \wp_{ba}}{\hbar}$$

single-photon Rabi frequency (or coupling constant)

Electric field of a a single photon

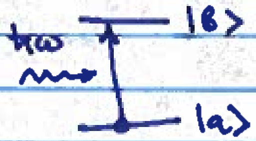
$$\langle b, n\pm 1|\hat{H}_{int}|a,n\rangle = -\hbar g_{ba}\begin{cases}\sqrt{n+1}\\ \sqrt{n}\end{cases}$$

Total $|a\rangle \to |b\rangle$ transition amplitude

$$c_b(t) + c_b'(t) = g_{ba}\left\{\sqrt{n}\,\frac{e^{i(\omega-\omega_{ba})t}-1}{\omega-\omega_{ba}} + \sqrt{n+1}\,\frac{e^{i(\omega+\omega_{ba})t}-1}{\omega+\omega_{ba}}\right\}$$

Near-resonant interaction $\omega \approx |\omega_{ba}| \to$ rotating wave approximation

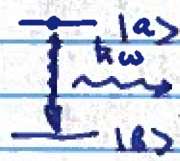
a) $E_b > E_a$ — the first term dominates



one photon is absorbed

$$P_{abs} = |c_b(t)|^2 \propto |g_{ba}|^2\cdot n$$

b) $E_b < E_a$ — the second term dominates



one photon is emitted

$$P_{em} = |c_b'(t)|^2 \propto |g_{ba}|^2\cdot(n+1)$$

$$\frac{P_{emission}}{P_{absorption}} = \frac{n+1}{n}$$

If $n=0$ (no light initially), the emission is still possible if $|a\rangle$ is an excited atomic state

$\to$ spontaneous emission

Let's rewrite the interaction Hamiltonian in a for ~~the~~ a two-level system

$$\hat{H} = \sum_{i=a,b} E_i |i\rangle\langle i| + \hbar\omega\left(\hat{a}^+ a + \frac{1}{2}\right) + \hbar\left(g_{ba}|b\rangle\langle a| + g_{ab}|a\rangle\langle b|\right) \times (\hat{a} + \hat{a}^+)$$

E_b For the atomic component $\hat{H}_a = E_a|a\rangle\langle a| + E_b|b\rangle\langle b|$

$E=0$ let's pick $E=0$ in the mid-point b/w $|a\rangle$ & $|b\rangle$

E_a

$\omega_0 = \frac{E_b - E_a}{2}$

$$\hat{H}_a = -\frac{E_b - E_a}{2}|a\rangle\langle a| + \frac{E_b - E_a}{2}|b\rangle\langle b| =$$

$$= \frac{1}{2}\hbar\omega_0\left(|b\rangle\langle b| - |a\rangle\langle a|\right) = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z$$

where $\hat{\sigma}_z = |b\rangle\langle b| - |a\rangle\langle a|$ inversion operator

$$\hat{H}_{int} = \left(\hbar g_{ab}|a\rangle\langle b| + \hbar g_{ba}|b\rangle\langle a|\right)(\hat{a} + \hat{a}^+) =$$

$$= \hbar g\,(\hat{\sigma}_- + \hat{\sigma}_+)(\hat{a} + \hat{a}^+)$$

where $\hat{\sigma}_+ = |a\rangle\langle b|$ and $\hat{\sigma}_+ = |b\rangle\langle a|$ are atomic transition operators

Three atomic operators $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ obey Pauli matrices commutation relationships

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z \quad \& \quad [\hat{\sigma}_z, \hat{\sigma}_\pm] = 2\sigma_\pm$$

The expectation values of these operators are related to atomic density matrix elements

$$\langle\psi|\sigma_+|\psi\rangle = \langle\psi|b\rangle\langle a|\psi\rangle = c_b^* c_a = \rho_{ab}$$

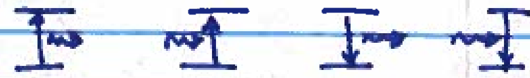
$$\langle\psi|\sigma_z|\psi\rangle = \langle\psi|b\rangle\langle b|\psi\rangle - \langle\psi|a\rangle\langle a|\psi\rangle = |b|^2 - |a|^2 = \rho_{bb} - \rho_{aa}$$

Using this notation

$$\hat{H}_{int} = \hbar g\,(\hat{\sigma}_- + \hat{\sigma}_+)(\hat{a} + \hat{a}^+)$$

$$\hat{H}_a = \frac{1}{2}\hbar\omega_0\,\hat{\sigma}_z$$

$$(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}^+ + \hat{a}) = \hat{\sigma}_+\hat{a}^+ + \hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^+ + \hat{\sigma}_-\hat{a}^-$$



For $\omega \approx \omega_0$ there are only two processes with non-vanishing probability:

① a photon is absorbed and the atom is excited

② a photon is emitted and the atom is de-excited

RWA Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega_0\,\hat{\sigma}_z + \hbar\omega\,\hat{a}^+\hat{a} + \hbar g\,(\hat{\sigma}_-\hat{a}^+ + \hat{\sigma}_+\hat{a})$$

If there is no interaction $(g=0) \rightarrow$ atomic and photonic components are decoupled

Eigenstate $|a,n\rangle$, $|b,n\rangle$ for any n

We will assume a closed system:

- an atom can only be found in two states $|a\rangle$ & $|b\rangle$ $\quad |a\rangle\langle a| + |b\rangle\langle b| = \hat{1}$
- the total energy of the system is conserved, so that the total number of excitations is constant

$$\hat{N}_e = |b\rangle\langle b| + \hat{a}^+\hat{a}$$

Two coupled states

—— $|1\rangle = |b,n\rangle$ $\qquad E^{(0)}_{1,n} = \frac{1}{2}\hbar\omega_0 + \hbar\omega n = \hbar\omega(n+\frac{1}{2}) + \frac{1}{2}\hbar(\omega_0-\omega)$

—— $|2\rangle = |a,n+1\rangle$ $\qquad E^{(0)}_{2,n} = -\frac{1}{2}\hbar\omega_0 + \hbar\omega(n+1) = \hbar\omega(n+\frac{1}{2}) - \frac{1}{2}\hbar(\omega_0-\omega)$

Energy splitting b/w $|1\rangle$ & $|2\rangle$ (in RWA)

$$\hbar\Delta = \hbar(\omega_0-\omega)$$

So now we are back to a two-~~photon~~ level system → but for a fully quantized atom-photon states

For a known photon-number states

$$\hat{H}_n = \hbar\omega\left(n+\tfrac{1}{2}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{2}\hbar\Delta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \hbar g\begin{pmatrix}0 & \sqrt{n+1}\\ \sqrt{n+1} & 0\end{pmatrix}$$

$$= \underbrace{\hbar\omega\left(n+\tfrac{1}{2}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\text{overall energy}} + \frac{1}{2}\hbar\begin{pmatrix}\Delta & 2g\sqrt{n+1}\\ 2g\sqrt{n+1} & -\Delta\end{pmatrix}$$

overall energy common for both states (not interesting) → neglect

Quantum Rabi flopping

$|\psi\rangle = c_a(t)\,|a, n+1\rangle + c_b(t)\,|b, n\rangle$

we will assume $c_b(0) = 1$, $c_a(0) = 0$

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

$$i\hbar\begin{pmatrix}\dot{c}_b\\ \dot{c}_a\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}\Delta & 2g\sqrt{n+1}\\ 2g\sqrt{n+1} & -\Delta\end{pmatrix}\begin{pmatrix}c_b\\ c_a\end{pmatrix}$$

~~This is~~ We have already encounter such problem for classical Rabi oscillations (with $2g\sqrt{n+1} \to \Omega$)

assuming $\Delta = 0$ for simplicity

$$c_b = \cos\left(g\sqrt{n+1}\,t\right)$$
$$c_a = -i\sin\left(g\sqrt{n+1}\,t\right)$$

$$|\psi(t)\rangle = \cos\left(g\sqrt{n+1}\,t\right)|b, n\rangle - i\sin\left(g\sqrt{n+1}\,t\right)|a, n+1\rangle$$

Quantum Rabi flopping

Note: even if $n=0$ (no photons), there still be oscillations @ frequency g → vacuum Rabi flopping

Thus, a Fock state with a fixed number of photons behaves very similar to a classical Rabi oscillations

What about a coherent state?

$$|\psi_{atom}\rangle\big|_{t=0} = |b\rangle\,, \quad |\psi_{light}\rangle\big|_{t=0} = \sum_{n=0}^{\infty} c_n |n\rangle = |\alpha\rangle$$

$$c_n = e^{-|\alpha|^2/2}\,\frac{|\alpha|^n}{\sqrt{n!}}$$

$$|\psi_{total}\rangle\big|_{t=0} = |b\rangle|\alpha\rangle$$

As we discussed before, the light atom interaction couples only pair of states $|b,n\rangle \leftrightarrow |a,n+1\rangle$, but now for all possible photon states $|n\rangle$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}\left(c_n \cos[g\sqrt{n+1}\,t]|b\rangle - i c_{n-1}\sin[g\sqrt{n}\,t]|a\rangle\right)|n\rangle$$

$$|\psi(t)\rangle = |\psi_a(t)\rangle + |\psi_b(t)\rangle$$

$$|\psi_a(t)\rangle = -i\sum_{n=0}^{\infty} c_n \sin(g\sqrt{n+1}\,t)\,|n+1\rangle$$

$$|\psi_b(t)\rangle = \sum_{n=0}^{\infty} c_n \cos(g\sqrt{n+1}\,t)\,|n\rangle$$

Average atomic inversion

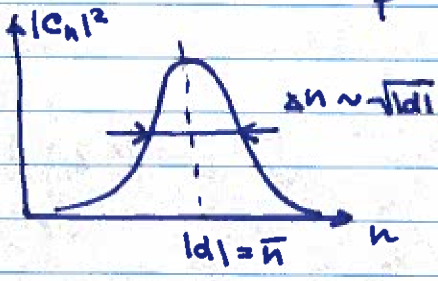
$$\langle\psi(t)|\,\hat{\sigma}_z\,|\psi(t)\rangle = \langle\psi(t)|\left(|b\rangle\langle b| - |a\rangle\langle a|\right)|\psi(t)\rangle =$$

$$= \langle\psi_b|\psi_b\rangle - \langle\psi_a|\psi_a\rangle = \sum_{n=0}^{\infty}|c_n|^2\left(\cos^2 g\sqrt{n+1}\,t - \sin^2 g\sqrt{n+1}\,t\right)$$

$$= \sum_{n=0}^{\infty}|c_n|^2 \cos 2g\sqrt{n+1}\,t = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\cos(2g\sqrt{n+1}\,t)$$

The output is a combination of many oscillations of some what different periods $\longrightarrow$ no clear oscillations

Coherent state photon number distribution



Main contributions comes from components with frequencies b/w $2g\sqrt{\bar{n}-\Delta n}$ and $2g\sqrt{\bar{n}+\Delta n}$

Corresponding phase spread

$$2gt_c\left(\sqrt{\bar{n}+\Delta n} - \sqrt{\bar{n}+\Delta n}\right) \approx 2gt_c\sqrt{\bar{n}}\left[\left(1+\frac{\Delta n}{2\bar{n}}\right)-\left(1-\frac{\Delta n}{2\bar{n}}\right)\right]$$

$$\approx 2gt_c \overbrace{\frac{\Delta n}{\sqrt{\bar{n}}}}^{\sim 1} \approx 2gt_c \quad \Rightarrow \quad t_c \sim 1/g$$

depends only on the coupling strength

However, we can also expect to see a revival of Rabi oscillations after some time t_R

$$\left(g\sqrt{\bar{n}+1} - g\sqrt{\bar{n}}\right)t_R = 2\pi \quad (\text{or in general } 2\pi k, k=1,2\ldots)$$

$$g\sqrt{\bar{n}}\left(1+\frac{1}{2\bar{n}}-1\right)t_R = \frac{gt_R}{2\sqrt{\bar{n}}} = 2\pi$$

$$t_R \approx \frac{4\pi\sqrt{\bar{n}}}{g}$$

The revival is never complete, since the frequiencies $\{g\sqrt{n}\}$ are not truly equidistant?

Why a coherent state is less "classical" than a number state?

Clear Rabi flopping require knowledge of precise intensity, that is provided by ~~the~~ a Fock state. A coherent state is a minimum uncertainty state, thus it has certain spread in its intensity distribution that leads to the Rabi flopping diffusion

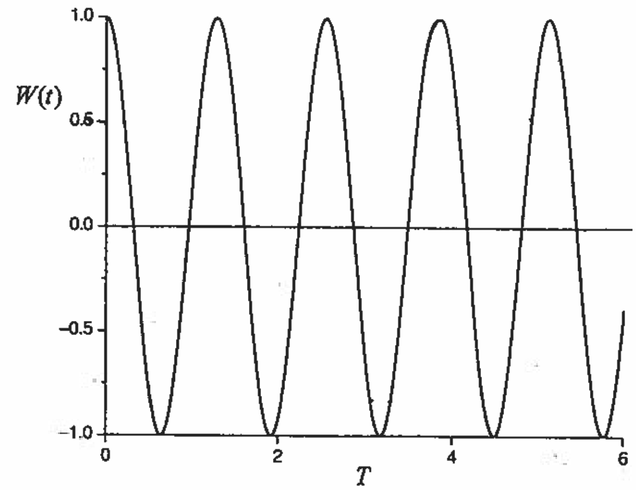


Fig. 4.6. Periodic atomic inversion with the field initially in a number state $|n\rangle$ with $n = 5$ photons.

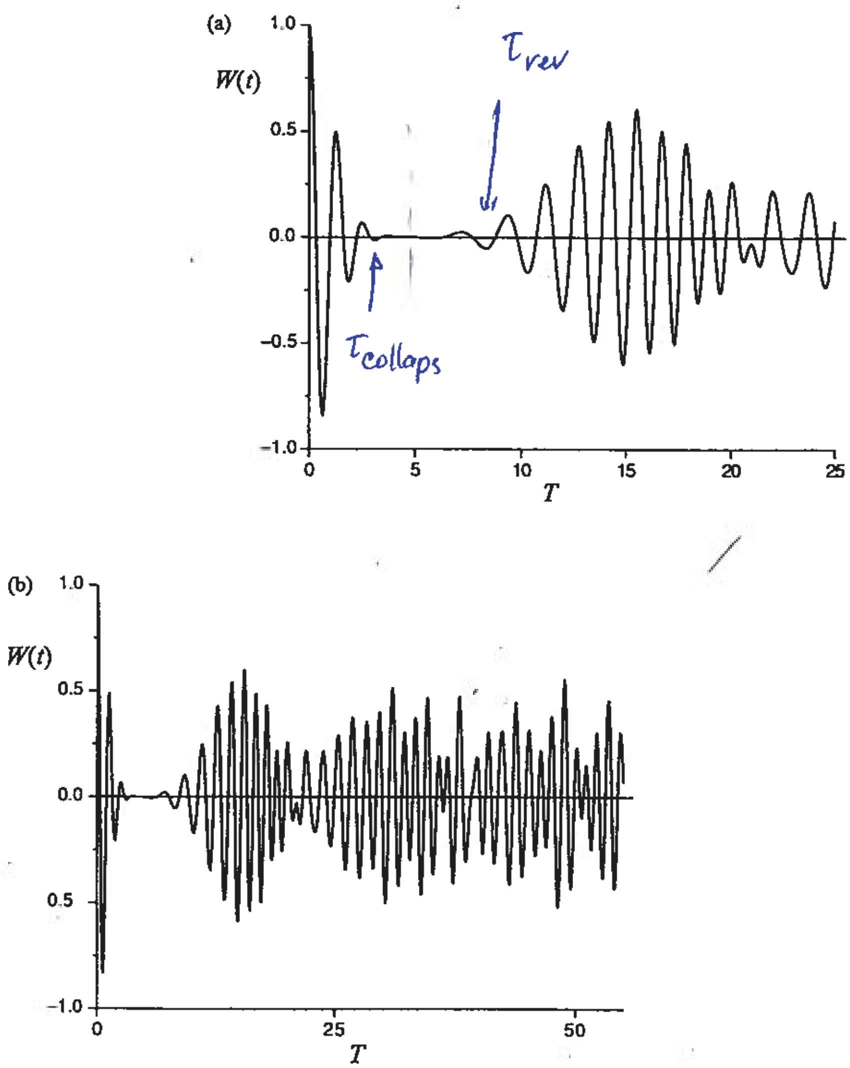


Fig. 4.7. (a) Atomic inversion with the field initially in a coherent state $\bar{n} = 5$. (b) Same as (a) but showing the evolution for a longer time, beyond the first revival. Here, T is the scaled time gt.