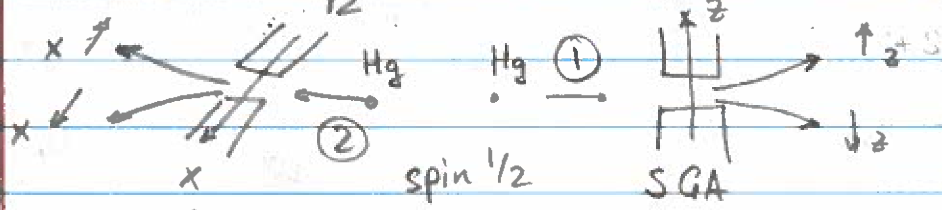
EPR paradox

Two-particle entangled state

$$|\psi^{(e)}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right) \qquad [\,|\psi^-\rangle \text{ Bell state}\,]$$



After a measurement

If $|\uparrow\rangle_z$ measured for 1 $\rightarrow$ $|\downarrow\rangle_z$ for 2

However, one can rewrite the state $|\psi^{(e)}\rangle$ in $|\pm\rangle_x$ basis

$$|\psi^{(e)}\rangle = \frac{1}{\sqrt{2}}\left(|+_1 -_2\rangle - |-\rangle_1|+\rangle_2\right)$$

If $|+\rangle_x$ measured for 1 $\rightarrow$ $|-\rangle_x$ for 2

Thus, the state of 2 seem to change depending on 1 measurements, even with no interactions between them — non-locality.

Wave-function formalism is unadequate since it seems that we cannot consistently describe the state of the system before measurements

Density matrix for particle 2

1 is measured in z-basis

$$\rho_2 = \langle\uparrow_1|\psi^{(e)}\rangle\langle\psi^{(e)}|\uparrow\rangle + \langle\downarrow_1|\psi^{(e)}\rangle\langle\psi^{(e)}|\downarrow\rangle =$$

$$= \frac{1}{2}\left(|\downarrow_2\rangle\langle\downarrow_2| + |\uparrow_2\rangle\langle\uparrow_2|\right) = \frac{1}{2}\begin{pmatrix} +1 & 0 \\ 0 & 1 \end{pmatrix}$$

if ① is measured in x-basis

$$\rho_2 = \langle +_1|\psi^{(2)}\rangle\langle\psi^{(2)}|+_1\rangle + \langle -_1|\psi^{(2)}\rangle\langle\psi^{(2)}|-_1\rangle =$$

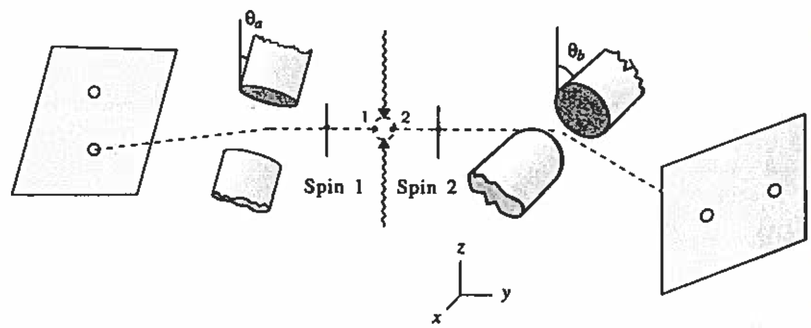
$$= \frac{1}{2}\left(|-_2\rangle\langle -_2| + |+\rangle_2\langle +|_2\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

same state for the particle ②

Entanglement raises the question of locality of the quantum mechanics — "spooky action at the distance"

Hidden variable theory → there is a parameter in each particle that controls the outcome of each measurement. Since we don't know its parameters, the outcome will see random to us, but is actually pre-set before particles part

Let's assume that the spin direction is pre-set, so any measurement results can be predicted beforehand.



We'll consider three possible measurement orientations : $\theta_a, \theta_b, \theta_c$; "+" - pass, "-" - block

Joint probability P_{ab} : ① passes through θ_a oriented detector, ② passes through θ_b oriented detector

$$P_{ab} = \left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\text{particle 1}} & \multicolumn{3}{c}{\text{particle 2}} \\ + & - & \bigcirc & - & + & \bigcirc \\ a & b & c & a & b & c \end{array} \right)$$

anti-correlated

Analogously, $P_{bc} = (\bigcirc + - \mid \bigcirc - +)$

$P_{ac} = (+ \bigcirc - \mid - \bigcirc +)$

Logically

$$P_{ab} = (+ - + \mid - + -) + (+ - - \mid - + +)$$

$$P_{bc} = (+ + - \mid - - +) + (- + - \mid + - +)$$

$$P_{ac} = (+ + - \mid - - +) + (+ - - \mid - + +)$$

$$P_{ab} + P_{bc} = \overbrace{(+ + - \mid - - +)} + (- + - \mid + - +) + (+ - + \mid - + -) + \overbrace{(+ - - \mid - + +)}$$

$$= P_{ac} + (+ + - \mid + - +) + (+ - + \mid - + -) \geq P_{ac}$$

$$P_{ab} + P_{bc} \geq P_{ac}$$ Bell's theorem inequality

Quantum calculation of the correlations in Bell's theorem

Spin rotation $|\theta\rangle = e^{-i\theta\hat{\sigma}_y}|\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$

Projection operator $\hat{\pi}_\theta = |\theta\rangle\langle\theta|$

such that $P_\psi(\theta) = \langle\psi|\theta\rangle\langle\theta|\psi\rangle$ – probability of a particle at the state $|\psi\rangle$ to pass the detector

$$\hat{\pi}_\theta = \left(\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle\right)\left(\cos\frac{\theta}{2}\langle\uparrow| + \sin\frac{\theta}{2}\langle\downarrow|\right) =$$

$$= \left[\cos^2\frac{\theta}{2}|\uparrow\rangle\langle\uparrow| + \sin^2\frac{\theta}{2}|\downarrow\rangle\langle\downarrow| + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\right]$$

$$= \frac{1}{2}\left(1 + \hat{\sigma}_z\cos\theta + \hat{\sigma}_x\sin\theta\right)$$

$$P_{ab} = \langle\psi^{(e)}|\hat{\pi}^{(1)}_{\theta_a}\hat{\pi}^{(2)}_{\theta_b}|\psi^{(e)}\rangle = \frac{1}{4}\left[1 - \cos(\theta_a - \theta_b)\right] =$$

$$= \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right)$$

Bell's inequality

$$\frac{1}{2}\sin^2\frac{(\theta_a - \theta_b)}{2} + \frac{1}{2}\sin^2\frac{(\theta_b - \theta_c)}{2} \geq \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_c}{2}\right)$$

For $\theta_a = 0$, $\theta_b = \pi/4$, $\theta_c = \pi/2$,

we must compare LHS: $\sin^2\pi/8 = \frac{2-\sqrt{2}}{4} \approx 0.15$

RHS: $\frac{1}{2}\sin^2\pi/4 = \frac{1}{4} = 0.25$

clearly Bell's inequality is violated!

Entanglement

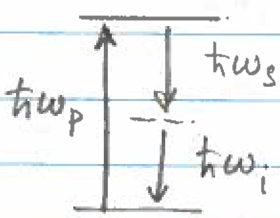
Two systems are placed in a quantum state that cannot be factorized into a product of individual systems.

Example: spontaneous emission

$$|\psi\rangle = e^{-\Gamma/2\, t}\,|b,0\rangle + \sum_i |a,1_i\rangle\, c$$

For $t \lesssim 1/\Gamma$ the states of an atom and of a spontaneous photons are entangled (i.e. making a measurement on the atomic state will provide information about photons)

Most common soures of entangled photons — parametric down conversion



1 pump photon (2ω) $\longrightarrow$

$\longrightarrow$ 1 signal + 1 idler photons (ω)

Nonlinear interaction

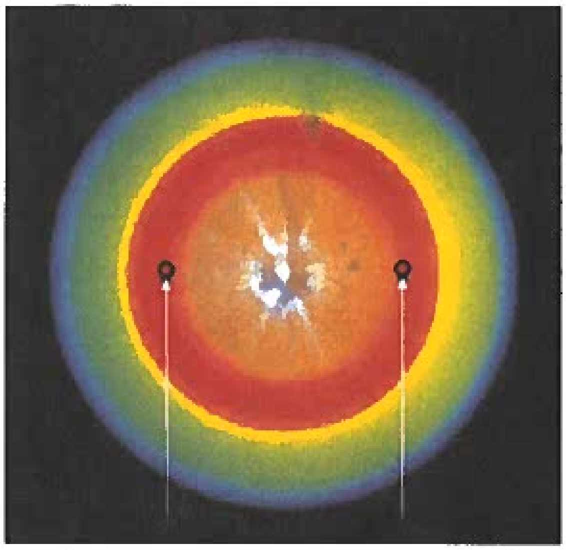
$$\hat{H}_{int} = \chi^{(2)}\, \hat{a}_p\, \hat{a}_s^+ \hat{a}_i^+ + c.c$$

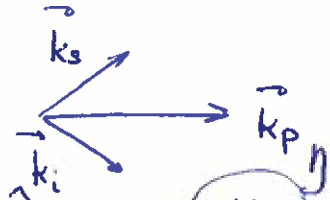
$$\simeq \chi^{(2)} \alpha_p\, \hat{a}_s^+ \hat{a}_i^+$$ (for strong coherent pump laser

Energy conservation: $\hbar\omega_p = \hbar\omega_s + \hbar\omega_i$

Momentum conservation $\hbar\vec{k}_p = \hbar\vec{k}_i + \hbar\vec{k}_s$

$\rightarrow$ Phase-matching conditions





$$\hat H_{int} = \chi^{(2)} \alpha_p \, \hat a_s^+ \hat a_i^+ + h.c$$

Initial ~~state~~ for signal & idler

$$|\psi_0\rangle \,\hat{=}\, |0\rangle_s |0\rangle_i$$

$$|\psi(t)\rangle = e^{-i\hat H_{int} t/\hbar} \, |\psi(t=0)\rangle \approx$$

$$\approx \left(1 - \frac{i\chi\alpha_p}{\hbar} t \, \hat a_s^+ \hat a_i^+ + \frac{1}{2}\left(-\frac{i\chi\alpha_p}{\hbar}\right)^2 t^2 (\hat a_s^+)^2 (\hat a_i^+)^2 + \dots \right) |0\rangle_s |0_i\rangle =$$

$$\frac{\chi^{(2)}\alpha_p}{\hbar} t \ll 1$$

$$\approx |0\rangle_s |0_i\rangle + \overbrace{\frac{i\chi^{(2)}\alpha_p}{\hbar} t}^{\mu} \, |1\rangle_s |1\rangle_i + \dots$$

$$\approx |0\rangle_s |0_i\rangle - \mu \, \underline{|1\rangle_s |1\rangle_i}$$

correlated ~~pai~~ photon pair

For weak pumping

$$|\psi(t)\rangle = e^{-i\hat{H}_{int} t/\hbar} |\psi(0)\rangle \qquad |\psi(0)\rangle = |0\rangle_s |0\rangle_i$$

$$|\psi(t)\rangle \underset{\mu \ll 1}{\approx} \left(1 - i\hat{H}_{int} t/\hbar + \frac{1}{2}\left(-it\hat{H}_{int}/\hbar\right)^2 \dots\right)|\psi(0)\rangle =$$

$$= \left(1 - \boxed{\frac{i\chi^{(2)} d_p}{\hbar}}\, t\, \hat{a}_s^+ \hat{a}_i^+ + \dots\right)|0\rangle_s |0_i\rangle \approx |0\rangle_s |0_i\rangle - i\mu |1\rangle_s |1_i\rangle$$

Source of correlated photons, but not an entangled state

However, one can use this process to generate entanglement, if we consider two possible polarizations

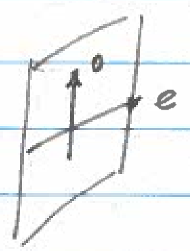
Because of the phase-matching conditions

$$\hbar \frac{n_p \omega_p}{c} \vec{e}_p = \hbar \frac{n_i \omega_i}{c} \vec{e}_i + \hbar \frac{n_s \omega_s}{c} \cdot \vec{e}_s$$

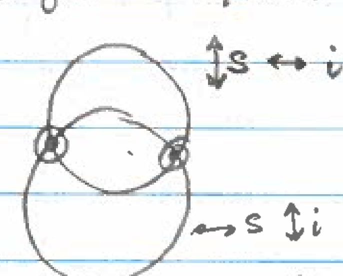
if the refractive indices n_i and n_s are different for different polarizations, they will phase-match at different angles.

Birefringent nonlinear crystals

$n_e \neq n_o$



single crystal



Type II down-conversion

~~Alternatively — two crystal arrangement~~

~~two overlapping cones with orthogonal polarizations~~

Type II down conversion

$$\hat{H}^{(2)}_{int} = \hbar g \left(\hat{a}^\dagger_{Vs}\hat{a}^\dagger_{Hi} + \hat{a}^\dagger_{Hs}\hat{a}^\dagger_{Vi}\right) + H.c.$$

$$|\psi^{(2)}(t)\rangle \approx |0\rangle_{Vs}|0\rangle_{Hs}|0\rangle_{Vi}|0\rangle_{Hi} -$$

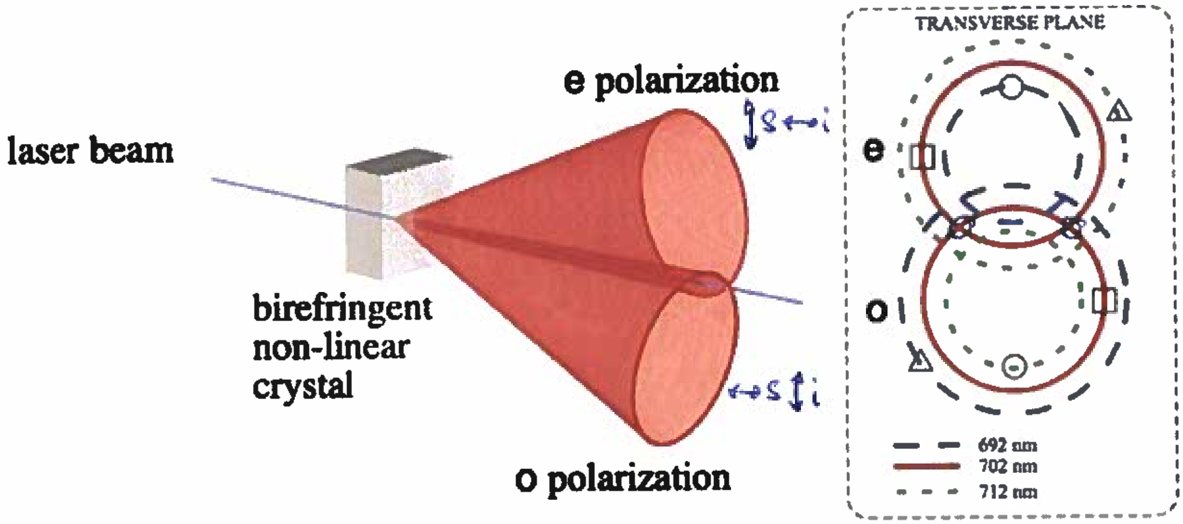
$$- i\mu \frac{1}{\sqrt{2}} \left(\underbrace{|1\rangle_{Vs}|0\rangle_{Hs}}_{|V\rangle_s}\underbrace{|0\rangle_{Vi}|1\rangle_{Hi}}_{|H\rangle_i} + \underbrace{|0\rangle_{Vs}|1\rangle_{Hs}}_{|H\rangle_s}\underbrace{|1\rangle_{Vi}|1\rangle_{Vs}}_{|V_i\rangle_i}\right)$$

$$|\psi^{(2)}(t)\rangle \approx |0\rangle_s|0\rangle_i - \frac{i\mu}{\sqrt{2}}\left(\underbrace{|V\rangle_s|H\rangle_i + |H\rangle_s|V\rangle_i}_{\text{polarization-entangled two-photon state}}\right)$$

One of Bell states

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle_s|H\rangle_i \pm |H_s\rangle|V_i\rangle\right)$$

$$|\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_s|H\rangle_i \pm |V\rangle_s|V\rangle_i\right)$$



$$\hat{H}_{int} = \hbar g\left(\hat{a}^{+}_{Vs}\,\hat{a}^{+}_{Hi} + \hat{a}^{+}_{Hs}\,\hat{a}^{+}_{Vi}\right) + H.c$$

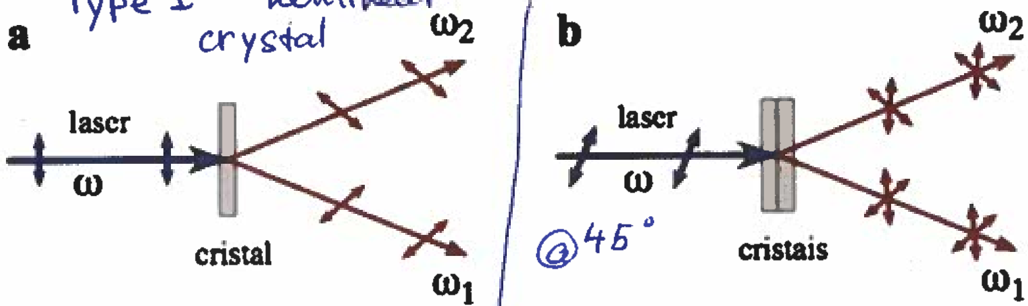
$$|\psi_0\rangle = |0\rangle_{sV}\,|0\rangle_{sH}\,|0\rangle_{iV}\,|0\rangle_{iH}$$

$$\cancel{H_i}\quad |\psi(t)\rangle = |0\rangle_{sV}|0\rangle_{sH}|0_{iV}\rangle|0_{iH}\rangle + \mu\frac{1}{\sqrt{2}}\Big(|1\rangle_{sV}|0\rangle_{sH}|0\rangle_{iV}|1\rangle_{iH} + |0\rangle_{sV}|1\rangle_{sH}|1\rangle_{iV}|0\rangle_{iH}\Big)$$

$$|0\rangle_{V}|1\rangle_{H} = |H\rangle$$
$$|1\rangle_{V}|0\rangle_{H} = |V\rangle$$

$$|\psi(t)\rangle = |0\rangle_{vac} + \frac{\mu}{\sqrt{2}}\underbrace{\left(|V\rangle_s|H\rangle_i + |H\rangle_s|V_i\rangle\right)}_{\text{entangled state}}$$

Type I - nonlinear crystal



@45°

$$|H\rangle_p \rightarrow |V\rangle_s |V\rangle_i$$

$$\hat{H}_{int} = \hbar g \, a^+_{Vs} a^+_{Vi} + h.c.$$

$$\hat{H}_{int} = \hbar g \left(\hat{a}^+_{Vs} \hat{a}^+_{Vi} + \hat{a}^+_{Hs} \hat{a}^+_{Hi} \right) + h.c.$$

$$|\psi(t)\rangle = |vac\rangle + \frac{1}{\sqrt{2}} \mu \left(|V\rangle_s |V\rangle_I + |H\rangle_s |H\rangle_I \right)$$

entangled state

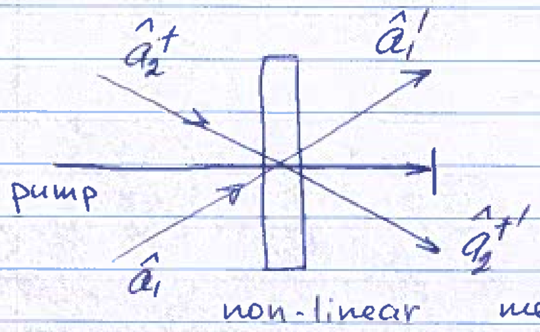
Bell states

eigen-basis for two-particles entangle states are

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|V\rangle_s |H\rangle_I \pm |H\rangle_s |V\rangle_I \right)$$

$$|\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|V\rangle_s |V\rangle_I \pm |H\rangle_s |H\rangle_I \right)$$

Optical parametric amplification



$\hat{a}_1, \hat{a}_2^+$ – input modes (can be a vacuum states)

$\hat{a}_1'\ \hat{a}_2^{+\prime}$ – output modes

Typical Hamiltonian $\hat{H}_{OPA} \propto \chi^{(3)}\ \hat{a}_p \hat{a}_1^+ \hat{a}_2^+ + \hat{a}_p^+ \hat{a}_1 \hat{a}_2$
replacing $\hat{a}_p \longrightarrow \alpha_p$ (strong coherent pump)

Solution: $\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2^{+\prime} \end{pmatrix} = A \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^+ \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{\text{Bogoliubov transformation}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^+ \end{pmatrix}$

Same ~~beha~~ behaviour ~~ca~~ can occure in case of other non-linear processes (i.e. four-wave mixing)

To preserve the commutation relationships for $\hat{a}_1$ and $\hat{a}_2$: $|A_{11}|^2 - |A_{12}|^2 = |A_{21}|^2 - |A_{22}|^2 = 1$
$A_{11}^* A_{21} - A_{12}^* A_{22} = 0$

That also implies $\hat{a}_1^{\prime +} \hat{a}_1' - \hat{a}_2^{+\prime} \hat{a}_2' = \hat{a}_1^+ \hat{a}_1 - \hat{a}_2^+ \hat{a}_2$
photon difference number is conserved
(photons are created in pairs)

Note that the total number of photons $\hat{a}_1^+ \hat{a}_1 + \hat{a}_2^+ \hat{a}_2$ is not conserved due to amplification

In case of real A_{ij} coefficients, one can present the transformation matrix A as

$$A = \begin{pmatrix} \cosh(\theta/2) & \sinh(\theta/2) \\ \sinh(\theta/2) & \cosh(\theta/2) \end{pmatrix}$$

[if A_{ij} are complex, it can be ~~so~~ included as two separate phase shifters before and after the amplification matrix)

Two quadratures are transformed in a similar way

Note: I will use q and p instead of X_1 and X_2 to avoid confusion with two channel $\quad q \equiv X_1 \;, \; p \equiv X_2$

$$\begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \begin{pmatrix} \cosh\theta/2 & \sinh\theta/2 \\ \sinh\theta/2 & \cosh\theta/2 \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix}$$

$$\begin{pmatrix} \hat{p}_1' \\ \hat{p}_2' \end{pmatrix} = \begin{pmatrix} \cosh\theta/2 & -\sinh\theta/2 \\ -\sinh\theta/2 & \cosh\theta/2 \end{pmatrix} \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix}$$

Using $\begin{pmatrix} \cosh\theta/2 & \sinh\theta/2 \\ \sinh\theta/2 & \cosh\theta/2 \end{pmatrix} = R^{-1} \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} R$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad 45^\circ \text{ rotation matrix}$$

$$\hat{R}\times \begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \overbrace{R^{-1}}^{\hat{R}\times} \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} R \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix}$$

$$\hat{R}\times \begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{q}_1' + \hat{q}_2' \\ \hat{q}_1' - \hat{q}_2' \end{pmatrix} = \begin{pmatrix} \hat{q}_+' \\ \hat{q}_-' \end{pmatrix}$$

$$\hat{q}_\pm' = \frac{\hat{q}_1' \pm \hat{q}_2'}{\sqrt{2}} \qquad \text{Rotated joint quadrature}$$

Thus, for the joint quadratures

$$\begin{pmatrix} \hat{q}_+' \\ \hat{q}_-' \end{pmatrix} = \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} \begin{pmatrix} \hat{q}_+ \\ \hat{q}_- \end{pmatrix}$$

or $\hat{q}_-' = e^{-\theta/2}\,\hat{q}_-$

For strong amplification $\theta \to \infty$ $e^{-\theta/2} \to 0$

thus $\hat{q}_-' \to 0$

$\hat{q}_1 - \hat{q}_2 \to 0$

Similarly $\hat{p}_1 + \hat{p}_2 \to 0$

Quadratures are strongly correlated, which is the indication of entanglement.

Thus, measurements of the first field allow predict with certainty the measurements for the second

since $\hat{q}_1 = \hat{q}_2$

and $\hat{p}_1 = -\hat{p}_2$

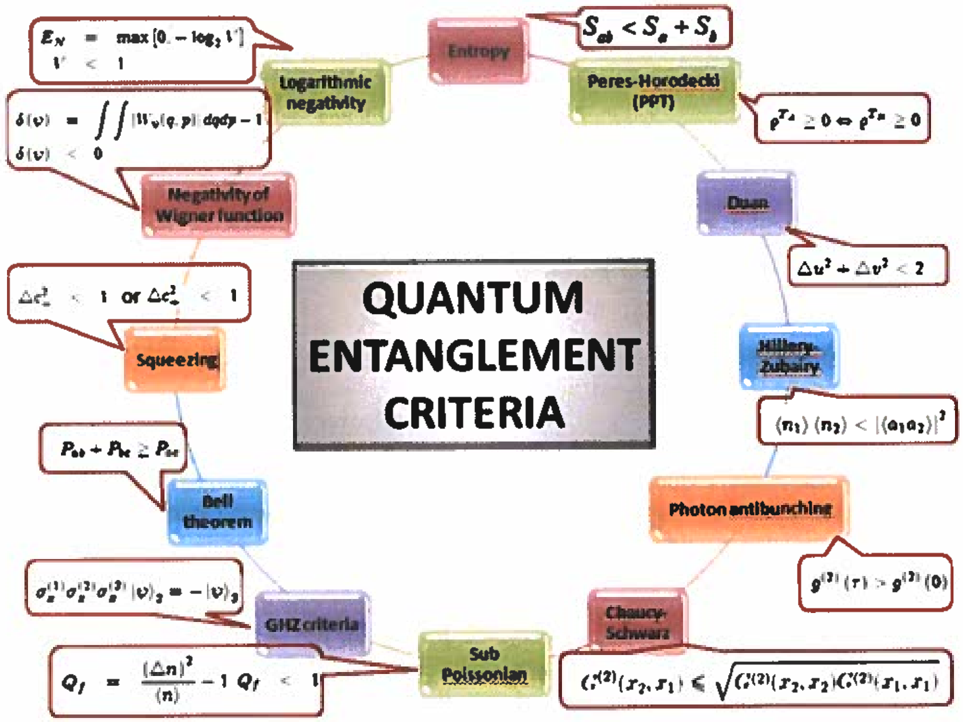


Figure 1. Various entanglement criteria widely used to detect entanglement.

CHECKLISTS PROPERTIES OF QUANTUM ENTANGLEMENT

Properties/ Criteria	Necessary condition	Sufficient condition	Density operator	No. of Photon	Correlation	Phase sensitive
Entropy	✓		✓			
Peres Horodecki	✓		✓			
Duan		✓		✓	✓	✓
Hillery-Zubairy		✓		✓	✓	✓
Antibunching	✓			✓	✓	✓
Chaucy-Schwarz	✓			✓	✓	
Sub Poissonian	✓			✓	✓	✓
GHZ		✓		✓		
Bell's theorem	✓			✓		
Squeezing	✓			✓	✓	✓
Negative Wigner function		✓	✓			✓
Logarithmic Negativity		✓		✓		✓

Figure 2. Properties of each entanglement criteria.