Mechanical Effects of Light

Laser cooling arises in a simple way from the conservation of momentum and energy. We will begin first with a heuristic treatment, due originally to David Wineland. Then we will use the density matrix method and the results of Chapter 2 to calculate the forces and viscous damping arising from *"Optical Molasses,"* the term coined by Steven Chu. These ideas will be used to understand the mechanics of a *Magneto-Optical Trap*, which is the work horse for initially producing cold atoms in most atom-cooling and trapping experiments.

3.1 Heuristic Treatment of Optical Cooling

We consider an atom, initially moving with velocity $\mathbf{v}$, which makes a transition from the ground state $|g\rangle$ to the excited state $|e\rangle$ by absorbing a photon, as shown in Fig. 3.1. We take $\hbar\omega_0 = E_e - E_g$. Consider first the resonance frequency for *absorption* of a photon, as shown in Fig. 3.2. The atom is initially in the ground state moving with a momentum $\mathbf{p} = m\mathbf{v}$. By absorbing a photon, the atom makes a transition to the excited state, moving with a new momentum $\mathbf{p}'$. We assume that the atom makes the transition by absorbing a photon of frequency ω_{abs} and energy $\hbar\omega_{abs}$, with a wave vector $\mathbf{q}$ and momentum $\hbar\mathbf{q}$. Then, momentum conservation requires

$$\mathbf{p}' = \mathbf{p} + \hbar\mathbf{q}.$$

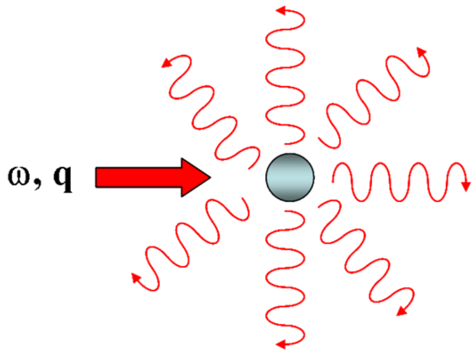

Figure 3.1: An atom absorbs laser photons of momentum $\hbar\mathbf{q}$ and then spontaneously emits photons in a random direction, producing an average momentum transfer along $\mathbf{q}$.

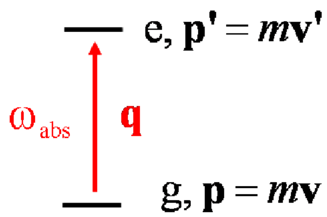

Figure 3.2: An atom, initially in the ground state g with velocity $\mathbf{v}$ absorbs a photon of momentum $\hbar\mathbf{q}$ in the direction of the laser beam. Note that $\hbar\omega_0 \equiv E_e - E_g$ for an atom at *rest*.

Energy conservation requires

$$\hbar\omega_{abs} + \frac{\mathbf{p}^2}{2m} + E_g = E_e + \frac{(\mathbf{p} + \hbar\mathbf{q})^2}{2m}.$$

Then,

$$\hbar(\omega_{abs} - \omega_0) = \frac{(\mathbf{p} + \hbar\mathbf{q})^2 - \mathbf{p}^2}{2m} = \frac{\mathbf{p}}{m} \cdot \hbar\mathbf{q} + \frac{\hbar^2\mathbf{q}^2}{2m}. \tag{3.1}$$

We define the recoil energy

$$E_R \equiv \frac{\hbar^2\mathbf{q}^2}{2m}. \tag{3.2}$$

This is the *kinetic* energy that would be imparted to an atom, initially at rest, when it absorbs a photon. For ^{6}Li, the mass $m = 10^{-23}$g, the resonant wavelength is $\lambda = 0.67\,\mu$m. In this case, $E_R/k_B = 3.5\,\mu$K, i.e., the energy corresponds to a the thermal energy $k_B T$ with $T = 3.5$ micro-Kelvin. Since $\mathbf{p} = m\mathbf{v}$, the resonance frequency for absorption of a photon is

$$\omega_{abs} = \omega_0 + \mathbf{v}\cdot\mathbf{q} + \frac{E_R}{\hbar}. \tag{3.3}$$

We can rewrite this in terms of the Doppler shifted frequency in the atom frame: $\omega_{abs} - \mathbf{v}\cdot\mathbf{q} = (\hbar\omega_0 + E_R)/\hbar$. This is just the total energy of the atom, relative to that of the ground state for an atom at rest, including the recoil kinetic energy. We see that recoil makes the resonant energy to absorb a photon larger than the atom rest frame value $\hbar\omega_0$.

Now consider the frequency for spontaneous *emission* of a photon[1]. As

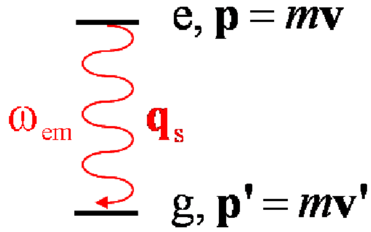

Figure 3.3: An atom, initially in the excited state e with velocity $\mathbf{v}$ spontaneously emits a photon of momentum $\hbar\mathbf{q}_s$ in a random direction.

shown in Fig. 3.3, an atom initially in state $|e\rangle$ and moving with velocity $\mathbf{v}$ spontaneously radiates a photon of frequency ω_{em} and wave vector $\mathbf{q}_s$. In this case, momentum conservation requires that the initial momentum of the atom equal the combined final momentum of the atom and the emitted photon: $\mathbf{p} = \mathbf{p}' + \hbar\mathbf{q}_s$. Hence, $\mathbf{p}' = \mathbf{p} - \hbar\mathbf{q}_s$. Energy conservation then requires

$$E_e + \frac{\mathbf{p}^2}{2m} = E_g + \hbar\omega_{em} + \frac{(\mathbf{p} - \hbar\mathbf{q}_s)^2}{2m}\,.$$

Then,

$$\omega_{em} = \omega_0 + \frac{\mathbf{p}^2 - (\mathbf{p} - \hbar\mathbf{q}_s)^2}{2m\hbar} = \omega_0 + \frac{\mathbf{p}}{m}\cdot\mathbf{q}_s - \frac{\hbar\mathbf{q}_s^2}{2m}.$$

[1]Here, we neglect stimulated emission, assuming that the intensity is low, so that $\rho_{11} \simeq 1$ and $\rho_{22} << 1$.

Hence,

$$\omega_{em} = \omega_0 + \mathbf{v} \cdot \mathbf{q}_s - \frac{E_R}{\hbar}. \tag{3.4}$$

Now, spontaneous photons are emitted symmetrically in space on average, as shown in Fig. 3.3, so that $\langle \mathbf{q}_s \rangle = 0$. Hence, for many absorption and emission cycles, the average spontaneous emission frequency is

$$\omega_{em} = \omega_0 - \frac{E_R}{\hbar}. \tag{3.5}$$

Power input to the atoms.

The net change in the mechanical energy of the atoms ΔE for an average absorption-emission cycle is just

$$\Delta E = \hbar\omega_{abs} - \hbar\omega_{em} = \hbar\mathbf{q} \cdot \mathbf{v} + 2\, E_R. \tag{3.6}$$

As shown in Chapter 2, the rate R of photon absorption can be written in terms of the optical cross section σ, $R = \sigma I/(\hbar\omega)$, where I is the laser intensity and we assume that $\omega \simeq \omega_0$, since the spontaneous linewidth and recoil frequency $E_R/\hbar$ are very small compared to ω_0. The mechanical *power* imparted to the atom is then

$$\frac{dE}{dt} = \frac{\sigma I}{\hbar\omega} \left(\hbar\mathbf{q} \cdot \mathbf{v} + 2E_R\right). \tag{3.7}$$

Note, since spontaneous emission is spatially symmetric and produces no net force on average, the average force on the atom is just

$$\langle \mathbf{F} \rangle = \frac{\sigma I}{\hbar\omega} \hbar\mathbf{q}.$$

This is just the absorption rate multiplied by the momentum per per photon. The first term in the power Eq. 3.7 is then just the scalar product of the average force with the velocity, as it should be. The second term in Eq. 3.7 arises from the recoil energy and describes the heating due to random walk in momentum space, as discussed further below.

Neglecting saturation, for small intensity, we have, from Chapter 2,

$$\sigma = \sigma(\Delta) = \sigma_0 \frac{\gamma_{21}^2}{\gamma_{21}^2 + \Delta^2}, \tag{3.8}$$

where $\Delta = \omega_{\text{atom frame}} - \omega_0$ is the detuning as observed in the rest frame of the atom. Here, $\gamma_{21} = \gamma_s/2$ and $\sigma_0 \equiv 3\lambda^2/(2\pi)$ is the optical cross section at resonance. Later, when we do a more detailed density matrix treatment, we will include saturation.

Optical Molasses

When a *single* laser beam is shined on an atom, the force can be very large (for an atom!). Near saturation, the optical rate $R \simeq 1/(2\tau_{spont})$, as we show below. For ^{6}Li, $\tau_{spont} = 27$ ns, and $R \simeq 2 \times 10^7$ per second. Then the force has a magnitude $F = R(h/\lambda)$, and the acceleration is $a = R\,h/(m\lambda)$. For ^{6}Li $h/(m\lambda) = 0.1$ m/s, so the corresponding acceleration is $2 \times 10^6\, m/s^2$ or 2×10^5 times the acceleration of gravity!

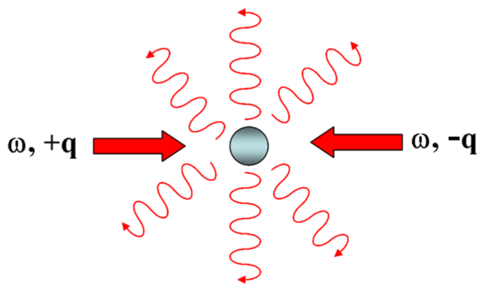

Figure 3.4: Two oppositely propagating beams of equal intensity with wave vectors $\pm\mathbf{q}$ produce no net force on an atom at rest. A damping force that opposes the velocity is produced when the laser frequency is tuned below resonance.

To avoid this rapid acceleration and to cool the atoms, a pair of counter-propagating laser beams are used, as shown in Fig. 3.4. The wave vectors for the beams are taken to be $\pm\mathbf{q}$, where $+$ denotes propagation along the positive x-axis, i.e., $\mathbf{q} = q\hat{\mathbf{e}}_x$. For equal intensities and equal detunings from resonance, $\omega - \omega_0$, the forces from these beams exactly cancel for an atom at *rest*, $\langle F\rangle_{total} = \langle F\rangle_{+\mathbf{q}} + \langle F\rangle_{-\mathbf{q}} = 0$.

Suppose instead that the atom is moving with a velocity $\mathbf{v}$. For the $+$ propagating beam, the detuning in the rest frame of the atom is $\Delta_+ = \omega - \mathbf{v}\cdot\mathbf{q} - \omega_0$, i.e., if the atom moves in the $+$ direction, the Doppler shift of the laser frequency is negative, while for the $-$ propagating beam, the corresponding detuning is positive $\Delta_- = \omega + \mathbf{v} \cdot \mathbf{q} - \omega_0$.

We cool the atoms by using a negative laser detuning, i.e., $\omega - \omega_0 < 0$. Then, there will be a force imbalance which tends to reduce the speed of a moving atom: When the atom moves *opposite* to the propagation direction of one beam, the Doppler shift increases the frequency in the rest frame of the atom, moving it closer to resonance, and increasing the backward force. The atom also is moving in the same direction as the second beam (which counterpropagates with respect to the first), which produces a downward Doppler shift, increasing the detuning in the rest frame of the atom, and moving it further from resonance. The net effect is that the atom is slowed by the force imbalance, which is always in a direction opposite to the component of the velocity along the x-axis. Using three sets of beams, along the x, y, and z directions, one obtains slowing in all three directions. This cools the atoms. Steven Chu coined the term "optical molasses" to describe this viscous damping effect.

To understand the cooling in more detail, we consider the net mechanical power imparted by both beams, using $\sigma(\Delta_\pm \equiv \omega - \omega_0 \mp \mathbf{v} \cdot \mathbf{q})$ for the respective optical cross sections. Then, for two beams of equal intensity I propagating along $\pm\mathbf{q}$,

$$\begin{aligned}
\frac{dE}{dt} &= \frac{I}{\hbar\omega}\left[\sigma(\Delta_+)(\hbar\mathbf{v} \cdot \mathbf{q} + 2E_R) + \sigma(\Delta_-)(-\hbar\mathbf{v} \cdot \mathbf{q} + 2E_R)\right] \\
&= \frac{\sigma_0 I}{\hbar\omega}\left\{\frac{(\gamma_s/2)^2(\hbar\mathbf{v} \cdot \mathbf{q} + 2E_R)}{(\omega - \omega_0 - \mathbf{v} \cdot \mathbf{q})^2 + (\gamma_s/2)^2}\right. \\
&\qquad \left. - \frac{(\gamma_s/2)^2(\hbar\mathbf{v} \cdot \mathbf{q} - 2E_R)}{(\omega - \omega_0 + \mathbf{v} \cdot \mathbf{q})^2 + (\gamma_s/2)^2}\right\}
\end{aligned} \tag{3.9}$$

We can easily show that Eq. 3.9 describes both viscous damping (when $\omega - \omega_0 < 0$) and heating, which arises from momentum diffusion. To see this, we expand the Lorentzian factors to lowest order in $\mathbf{q} \cdot \mathbf{v}$. We assume that for sufficiently small velocity and nonzero detuning $\mathbf{q} \cdot \mathbf{v} << \omega - \omega_0 \simeq$ few γ_s, i.e., the Doppler shifts are small compared to the linewidth for cold atoms. Then,

$$\begin{aligned}
\frac{1}{(\omega - \omega_0 \mp \mathbf{q} \cdot \mathbf{v})^2 + (\gamma_s/2)^2} &\simeq \frac{1}{(\omega - \omega_0)^2 + (\gamma_s/2)^2} \\
&\quad + \frac{\pm 2(\omega - \omega_0)\mathbf{q} \cdot \mathbf{v}}{[(\omega - \omega_0)^2 + (\gamma_s/2)^2]^2}.
\end{aligned} \tag{3.10}$$

Now, in Eq. 3.9, we see that the E_R terms are of the same sign and add, so that we retain only the first term in Eq. 3.10 to determine the lowest order E_R contribution. In contrast, the $\mathbf{q}\cdot\mathbf{v}$ terms in Eq. 3.9 are of opposite sign, so that the first term in Eq. 3.10 does not contribute. The second term in Eq. 3.10 produces two contributions in Eq. 3.9. These are $\propto (\mathbf{q}\cdot\mathbf{v})^2$ and add. Hence, we obtain

$$\frac{dE}{dt} = \frac{\sigma_0 I}{\hbar\omega}\frac{(\gamma_s/2)^2}{(\omega-\omega_0)^2+(\gamma_s/2)^2}\left\{\frac{2(\omega-\omega_0)\cdot 2\hbar(\mathbf{q}\cdot\mathbf{v})^2}{(\omega-\omega_0)^2+(\gamma_s/2)^2} + 4E_R\right\}. \tag{3.11}$$

We see that the second term in Eq. 3.11 always heats. It arises from absorption followed by spontaneous emission in a random direction, which causes momentum diffusion. The first term is negative, and cools the atoms if the detuning $\omega-\omega_0$ is negative. Note that the average cooling power for atoms moving along x (recall $\mathbf{q} = q\hat{\mathbf{e}}_x$) is $v_x\, F_{x-viscous}$. Hence, the first term in Eq. 3.11 shows that the viscous force must be of the form $-\alpha v_x$, so that the cooling power is of the form $-\alpha v_x^2$, where $\alpha > 0$ for $\omega-\omega_0 < 0$.

The minimum temperature in the Doppler limit.

The balance between the cooling rate and the heating rate determines the minimum temperature. As the friction force arises from the Doppler shift, the minimum temperature in this case is called the *Doppler* limit, which occurs for negligible saturation, as assumed in the above discussion. Let $\delta \equiv (\omega-\omega_0)/(\gamma_s/2)$ be the detuning in units of the linewidth, where $\gamma_s/2$ is the half width at half maximum (HWHM) of the Lorentzian factors. The cooling is maximized relative to the heating when the first term in brackets is as large as possible. Since

$$\frac{(\omega-\omega_0)}{(\omega-\omega_0)^2+(\gamma_s/2)^2} = \frac{2}{\gamma_s}\frac{\delta}{\delta^2+1},$$

we just maximize the δ-dependent factor by differentiation, and obtain $\delta^2 = 1$. Since we want negative detuning, the ideal case is $\delta = -1$ or

$$(\omega-\omega_0)_{opt} = -\frac{\gamma_s}{2}. \tag{3.12}$$

Using this in Eq. 3.11, we have

$$\left(\frac{dE}{dt}\right)_{opt} = \frac{\sigma_0 I}{\hbar\omega}\cdot 2\cdot\left\{-\frac{\hbar q^2 v_x^2}{\gamma_s} + E_R\right\}. \tag{3.13}$$

At equilibrium, when the average heating rate balances the cooling rate, the mean square velocity must obey

$$\frac{\hbar q^2}{\gamma_s}\langle v_x^2\rangle = E_R = \frac{\hbar^2 q^2}{2m}, \tag{3.14}$$

where we have used Eq. 3.2 for the recoil energy E_R. Hence, we obtain $\langle v_x^2\rangle = \hbar\gamma_s/(2m)$, or

$$\frac{m}{2}\langle v_x^2\rangle = \frac{\hbar\gamma_s}{4} \equiv \frac{k_B T_{Doppler}}{2}, \tag{3.15}$$

where we use the equipartition theorem for the energy in the x-direction. The Doppler limited temperature is then

$$\boxed{k_B T_{Doppler} = \frac{\hbar\gamma_s}{2}.} \tag{3.16}$$

For 3-dimensional cooling in the simplest configuration, we use 3 orthogonal sets of counter-propagating beam pairs to cool in 3 dimensions, which yields the same limiting temperature in each direction. For ^{6}Li, we have $\gamma_s = 2\pi \times 5.9$ MHz. Using $\hbar\gamma_s = h\gamma_s(Hz)$, we find $T_{Doppler} = 140\,\mu$K.

3.2 Density Matrix Treatment of Optical Forces

We already have all of the machinery in place to calculate the average force on an atom due to radiation pressure, including saturation. We will treat optical molasses and the magneto-optical trap in more detail, to expand on the heuristic treatment. We recall from Chapter 2 that the electric dipole interaction for a plane wave traveling in the z-direction is

$$V = -\vec{\mu}\cdot\mathbf{E} = -\vec{\mu}\cdot\left[\frac{\vec{\mathcal{E}}}{2}\,e^{ikz-i\omega t} + c.c\right]. \tag{3.17}$$

Here z is the location of the center of mass of the atom, and $\vec{\mu}$ is the electric dipole operator, which depends on the electron-nucleus relative coordinate. The operator describing the z-component of the force on the atom is

$$\mathbf{F}_{CM} = -\hat{\mathbf{e}}_z\frac{\partial}{\partial z}V = \hat{\mathbf{e}}_z\,\vec{\mu}\cdot\frac{\partial}{\partial z}\left[\frac{\vec{\mathcal{E}}}{2}\,e^{ikz-i\omega t} + c.c\right]. \tag{3.18}$$

Note that the dot product is between $\vec{\mu}$ and $\vec{\mathcal{E}}$. Since $\partial_z e^{\pm ikz} = \pm ik\, e^{\pm ikz}$, we have for the force operator

$$\mathbf{F} = ik\,\hat{\mathbf{e}}_z\,\vec{\mu}\cdot\left(\frac{\vec{\mathcal{E}}_0}{2}\,e^{-i\omega t} - \frac{\vec{\mathcal{E}}_0^*}{2}\,e^{i\omega t}\right), \tag{3.19}$$

where $\vec{\mathcal{E}}_0 \equiv \vec{\mathcal{E}}\,e^{ikz}$. The primary force is along $\hat{\mathbf{e}}_z$ due to k.

For a two-level atom, the quantum-averaged force is $\langle\mathbf{F}\rangle = Tr\{\rho\mathbf{F}\} = \rho_{12}\mathbf{F}_{21} + \rho_{21}\mathbf{F}_{12}$. We are interested in the time-averaged force (over a few optical cycles), so that we can eliminate terms that oscillate at 2ω. Then, we can write

$$\langle\mathbf{F}\rangle = 2\,Re\{\langle\rho_{21}\mathbf{F}_{12}\rangle_T\}, \tag{3.20}$$

where $\langle...\rangle_T$ denotes a time T average, as in Chapter 1 and

$$\mathbf{F}_{12} = ik\,\hat{\mathbf{e}}_z\,\vec{\mu}_{12}\cdot\left(\frac{\vec{\mathcal{E}}_0}{2}\,e^{-i\omega t} - \frac{\vec{\mathcal{E}}_0^*}{2}\,e^{i\omega t}\right), \tag{3.21}$$

From Chapter 2,

$$\rho_{21} = \lambda_{21}^-\,e^{-i\omega t}; \qquad \lambda_{21}^- = \frac{i}{\hbar}\frac{\vec{\mu}_{21}\cdot\vec{\mathcal{E}}_0}{2}\frac{\rho_{11}-\rho_{22}}{\gamma_s/2 - i\Delta}, \tag{3.22}$$

where Δ is the detuning in the rest frame of the atom. Then,

$$\langle\mathbf{F}\rangle = 2\,Re\left\{\left\langle\lambda_{21}^-\,e^{-i\omega t}(-ik\hat{\mathbf{e}}_z)\frac{\vec{\mu}_{12}\cdot\vec{\mathcal{E}}_0^*}{2}\,e^{i\omega t}\right\rangle_T\right\}, \tag{3.23}$$

where we have dropped terms $\propto \langle e^{-2i\omega t}\rangle_T \to 0$. Then, the force takes the simple form

$$\langle\mathbf{F}\rangle = Re\left\{-i\lambda_{21}^-\,\vec{\mu}_{12}\cdot\vec{\mathcal{E}}_0^*\right\}k\hat{\mathbf{e}}_z. \tag{3.24}$$

Using Eq. 3.22 and the Rabi frequency $\Omega \equiv \vec{\mu}_{21}\cdot\vec{\mathcal{E}}_0/\hbar$, we can write

$$\langle\mathbf{F}\rangle = Re\left\{\frac{\Omega^2}{2}\frac{\rho_{11}-\rho_{22}}{\gamma_s/2 - i\Delta}\right\}\hbar k\hat{\mathbf{e}}_z. \tag{3.25}$$

Repeating the same steps as used to obtain the transition rate $R = \sigma(\Delta)I/(\hbar\omega)$ in Chapter 2, we can rewrite Eq. 3.25 in the physically intuitive form

$$\langle\mathbf{F}\rangle = R(\rho_{11}-\rho_{22})\hbar k\hat{\mathbf{e}}_z. \tag{3.26}$$

Eq. 3.26 shows that the force is along +z when the ground state absorbs a photon by stimulated absorption at a rate R, and is along -z when the excited atom emits a photon by stimulated emission at a rate R. Each absorption (emission) imparts to the atom a momentum $\hbar k$ ($-\hbar k$). From Chapter 2, we recall that $\rho_{11} - \rho_{22} = \gamma_s/(\gamma_s + 2R)$. Then, we easily obtain

$$\langle \mathbf{F} \rangle = \frac{\gamma_s}{2} \hbar k \hat{\mathbf{e}}_z \frac{2R/\gamma_s}{1 + 2R/\gamma_s}. \tag{3.27}$$

Using the saturation intensity, $I_{sat} = \gamma_s \hbar\omega/(2\sigma_{opt})$, and the optical cross section $\sigma(\Delta) = \sigma_{opt}/(1 + \delta^2)$ from Chapter 2, we have

$$\frac{2R}{\gamma_s} = \frac{I}{I_{sat}} \frac{1}{1 + \delta^2}, \tag{3.28}$$

where $\delta = 2\Delta/\gamma_s$ is the detuning in units of the linewidth (HWHM), as defined above. Then,

$$\frac{2R/\gamma_s}{1 + 2R/\gamma_s} = \frac{I/I_{sat}}{1 + \delta^2 + I/I_{sat}}. \tag{3.29}$$

Finally, we obtain the force on the atom arising from a single beam of intensity I including saturation,

$$\boxed{\langle \mathbf{F} \rangle = \frac{\gamma_s}{2} \hbar k \hat{\mathbf{e}}_z \frac{I/I_{sat}}{1 + \delta^2 + I/I_{sat}}.} \tag{3.30}$$

From Eq. 3.30, we see that for high intensity, $I >> I_{sat}$, the maximum force is $\langle \mathbf{F} \rangle_{max} = \frac{\gamma_s}{2} \hbar k \hat{\mathbf{e}}_z$. This result has a simple interpretation. At high intensity, half of the atoms are in the excited state, so that $\gamma_s/2$ is the largest rate at which the atom can absorb a photon, recoil forward, and then spontaneously emit in a random direction. If the atom does not spontaneously emit, it will emit a photon back into the beam and recoil backward by stimulated emission, which cancels the recoil due to absorption. Hence, the spontaneous emission rate from the excited state determines the net number of forward recoils per second. As the photon momentum is $\hbar k$, the net force has a maximum of $\gamma_s \hbar k/2$.

In is instructive to determine all of the parameters for a ^{6}Li atom, where the resonant wavelength is $\lambda_0 = 0.67\,\mu$m and the mass is $m = 1.0 \times 10^{-23}$g.

The spontaneous lifetime is $\tau_{sp} = 27$ ns, so that the radiative decay rate $\gamma_s = 1/\tau_{sp} = 2\pi \times 5.9$ MHz, i.e., for weak laser beams, the absorption linewidth in the rest frame of the atom is 5.9 MHz, full-width at half maximum (FWHM). Then, $\hbar\omega = hc/\lambda_0 = 3.0 \times 10^{-12}$ erg. The resonant absorption cross section is $\sigma_{opt} = 3\lambda_0^2/(2\pi) = 2.1 \times 10^{-9}$ cm^2. Then, $I_{sat} = (\gamma_s/2)\hbar\omega/\sigma_{opt} = 2.6 \times 10^4$ ergs/s/cm^2= 2.6 mW/cm^2.

The maximum acceleration is obtained for $\omega = \omega_0$ and $I >> I_{sat}$: $a_{max} = F_{max}/m = \gamma_s\hbar k/(2m) = \gamma_s\, h/(2m\lambda_0) = 1.8 \times 10^8$ cm/s^2= 2×10^5 g's! Note that the large acceleration arises because the velocity change for absorption of a *single* photon is $\Delta v = h/(m\lambda_0) = 0.1$ m/s, and the maximum absorption rate $\gamma_s/2 = 1/(2\tau_{sp})$ is $\simeq 20$ million photons per second!

Optical Molasses and Viscous Damping

As shown in Fig. 3.5, for an atom moving to the right with velocity v, the Doppler shifts for two oppositely propagating beams are negative for the right-going beam and positive for the left going beam.

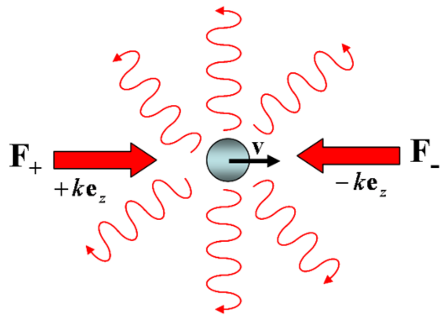

Figure 3.5: Two oppositely propagating beams of equal intensity with wave vectors $\pm k\hat{\mathbf{e}}_z$ produce no net force on an atom at rest. A damping force that opposes the velocity is produced when the laser frequency is tuned below resonance. For an atom moving to the right with velocity v, the Doppler shifted detunings for the right and left propagating beams are $\Delta_\pm = \Delta_0 \mp kv$, respectively, where $\Delta_0 = \omega - \omega_0$.

The net force on the moving atom, including saturation is obtained from

Eq. 3.30 with $\Delta \to \Delta_\pm$,

$$\begin{aligned}\langle \mathbf{F} \rangle &= \langle \mathbf{F}_+ \rangle + \langle \mathbf{F}_- \rangle \\ &= \frac{\gamma_s}{2}\hbar k \hat{\mathbf{e}}_z \left\{ \frac{I/I_{sat}}{1 + \frac{(\Delta_0 - kv)^2}{\gamma_{21}^2} + I/I_{sat}} - \frac{I/I_{sat}}{1 + \frac{(\Delta_0 + kv)^2}{\gamma_{21}^2} + I/I_{sat}} \right\}. \end{aligned} \quad (3.31)$$

For $I_\pm = I$ and $v = 0$, $\langle \mathbf{F} \rangle = 0$ for any detuning $\Delta_0 = \omega - \omega_0$. As in the heuristic treatment, we can expend the force to lowest order in the Doppler shift kv, assuming that for small v, the Doppler shift is small compared to the power broadened linewidth. Then,

$$\langle \mathbf{F} \rangle = v \left(\frac{\partial \langle \mathbf{F} \rangle}{\partial v} \right)_{v=0} = \sum_\pm \left(\frac{\partial \langle \mathbf{F}_\pm \rangle}{\partial \Delta_\pm} \right)_{v=0} \frac{\partial \Delta_\pm}{\partial v} v,$$

where the Doppler shift of the detunings $\Delta_\pm = \Delta_0 \mp kv$ makes the force velocity dependent.

$$\frac{\partial}{\partial v} \left. \frac{1}{1 + \frac{(\Delta_0 \mp kv)^2}{\gamma_{21}^2} + I/I_{sat}} \right|_{v=0} = \frac{-1}{\left[1 + \frac{\Delta_0^2}{\gamma_{21}^2} + I/I_{sat}\right]^2} \frac{2\Delta_0}{\gamma_{21}^2} (\mp k). \quad (3.32)$$

Hence, the net force is

$$\langle \mathbf{F} \rangle = \frac{-2\Delta_0/\gamma_{21}^2}{\left[1 + \frac{\Delta_0^2}{\gamma_{21}^2} + I/I_{sat}\right]^2} \frac{I}{I_{sat}} (-2kv) \frac{\gamma_s}{2} \hbar k \hat{\mathbf{e}}_z. \quad (3.33)$$

This result can be written more compactly using $\delta = \Delta_0/\gamma_{21}$ and $I' \equiv I/I_{sat}$.

$$\langle \mathbf{F} \rangle = \frac{\gamma_s}{2} \hbar k \hat{\mathbf{e}}_z \frac{\delta I'}{(1 + \delta^2 + I')^2} \frac{4k}{\gamma_{21}} v$$

Then, with $\gamma_{21} = \gamma_s/2$, we have

$$\langle \mathbf{F} \rangle \equiv -\alpha \, v \, \hat{\mathbf{e}}_z, \quad (3.34)$$

where α is the *viscous damping coefficient*, which is given by

$$\alpha = -4\hbar \, k^2 \frac{\delta I'}{(1 + \delta^2 + I')^2}. \quad (3.35)$$

Note that $\alpha > 0$ for $\delta < 0$, i.e., the force in Eq. 3.34 is a damping (cooling) force for $\delta < 0$.

The maximum damping coefficient, and hence maximum cooling is obtained for an optimum detuning δ_{opt} and an optimum intensity, i.e., $I_{opt}/I_{sat} = I'_{opt}$. To find the optimum values, we first assume that I' is given and find the optimum detuning. This requires $\partial\alpha/\partial\delta = 0$, or

$$\frac{I'}{(1+\delta^2+I')^2} - \frac{2\delta I' \cdot 2\delta}{(1+\delta^2+I')^3} = 0.$$

Then,

$$1 - \frac{4\delta^2}{1+\delta^2+I'} = 0,$$

so that $3\delta^2 = 1 + I'$. Hence,

$$\delta_{opt} = -\sqrt{\frac{1+I'}{3}}, \tag{3.36}$$

where we use the negative value to obtain damping. From Eq. 3.35 and Eq. 3.36, we then have for a given intensity,

$$\alpha_{opt}(I') = 4\hbar k^2 \, \frac{I'\sqrt{\frac{1+I'}{3}}}{\left[\frac{4}{3}(1+I')\right]^2}. \tag{3.37}$$

Next, we find the optimum intensity using $\partial\alpha_{opt}(I')/\partial I' = 0$. Hence, we require

$$\frac{\partial}{\partial I'}\left\{\frac{I'\sqrt{1+I'}}{(1+I')^2}\right\} = 0,$$

where we are assuming that δ_{opt} is adjusted for each I'. Then,

$$\frac{\partial}{\partial I'}\frac{I'}{(1+I')^{3/2}} = \frac{1}{(1+I')^{3/2}} - \frac{3}{2}\frac{I'}{(1+I')^{5/2}} = 0.$$

Then, $1 + I' = 3I'/2$, or $I'_{opt} = 2$, i.e., $I_{opt} = 2I_{sat}$. Then, Eq. 3.36 shows that $\delta_{opt} = -1$, i.e., $\Delta_0 = \omega - \omega_0 = -\gamma_s/2$. The optimum damping coefficient from Eq. 3.37 is then

$$\alpha_{opt} = 4\hbar k^2 \frac{2\sqrt{\frac{1+2}{3}}}{\left[\frac{4}{3}(1+2)\right]^2} = \frac{\hbar k^2}{2}.$$

Summarizing, for optimum damping, the detuning, laser intensity, and viscous damping coefficient are given by

$$\boxed{\begin{array}{l} \Delta_{opt} = -\dfrac{\gamma_s}{2}; \quad I_{opt} = 2\, I_{sat} \\[2ex] \qquad \alpha_{opt} = \dfrac{\hbar k^2}{2}. \end{array}} \tag{3.38}$$

The approximate size of the damping coefficient has a simple physical interpretation. The maximum force is of order $\gamma_s\, \hbar k/2$. As the forces from the opposing beams cancel for $v = 0$, the detuning arising from the Doppler shift for $v \neq 0$ determines the fraction of the maximum force that opposes the motion. The fraction of the total force is the order of the Doppler detuning divided by the linewidth, i.e., kv/γ_s, so that $F_{damp} \simeq (\gamma_s\, \hbar k/2)(kv/\gamma_s) \simeq \hbar k^2/2$. It is interesting to note that the linewidth cancels out.

Magneto Optical Trap

The damping force cools the atoms and produces confinement in velocity space. To build a *magneto-optical trap* or MOT, we need confinement in *real* space. This is accomplished by using counter-propagating beams of *opposite* circular polarization and a magnetic field gradient, generated with *opposing* magnet coils, as shown in Fig. 3.6.

The magnetic field tunes the energy levels of the atom, Fig. 3.7. As shown in Fig. 3.6, an atom at position $z > 0$ is in a magnetic field with $B_z > 0$. For a state with a negative g-factor, the magnetic moment is antiparallel to the angular momentum, the Zeeman energy shift is $-\vec{\mu}_{mag} \cdot \vec{B} = \mu_{Mag} B_z\, M$. When the detuning is negative as shown in Fig. 3.7, the local magnetic field causes the $M = +1$ level to Zeeman tune upward for $B_z > 0$, further from resonance with the right propagating σ_+ beam, while the $M = -1$ level tunes downward, closer to resonance with the left propagating σ_- beam. Hence, the net force is along $-z$. For an atom with $z < 0$, the situation is reversed, and the net force is along $+z$.

For the configuration shown in Fig. 3.6, since $\nabla \cdot \mathbf{B} = 0$, cylindrical symmetry requires $\partial B_x/\partial x = \partial B_y/\partial y = -(1/2)\partial B_z/\partial z$. Hence, to obtain three confinement in three directions, the handedness of the circular polarization is reversed in the x and y directions, relative to that of the z-direction.

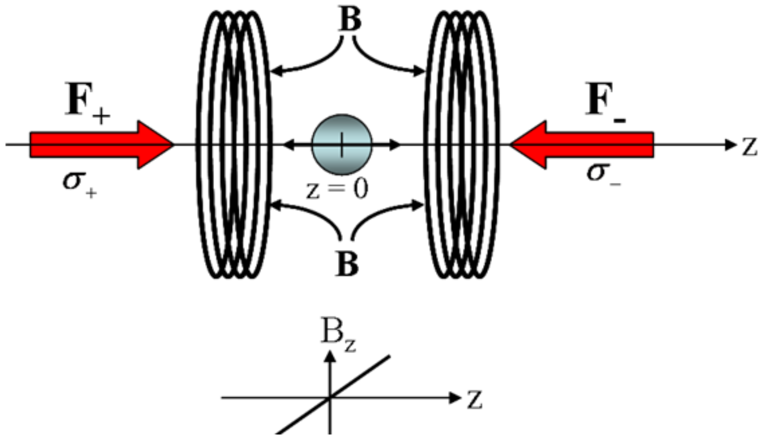

Figure 3.6: Spatial confinement is produced by Zeeman tuning a three-level atom with a spatially varying magnetic field. Two oppositely propagating beams of equal intensity with wave vectors $\pm k\hat{\mathbf{e}}_z$ and opposite circular polarization produce no net force on an atom at $z = 0$. A pair of opposing magnet coils produces a magnetic field gradient that Zeeman tunes the energy levels of the atom such that for $z \neq 0$ a restoring force is produced. For the configuration shown here, $\partial B_z/\partial z > 0$.

We see that for $z = 0$, $\langle \mathbf{F} \rangle = \langle \mathbf{F}_+ \rangle + \langle \mathbf{F}_- \rangle = 0$. We now show that the net force from the two opposing beams is proportional to $-z$, i.e., it is restoring, when the Zeeman tuning is small. To see this, note that the detunings of the two beams for an atom at position z (neglecting the Doppler shifts, which are already included in the damping force) are

$$\Delta_0^\pm = \omega - \omega_o^\pm = \omega - \omega_0 \mp \frac{\mu}{\hbar}\frac{\partial B_z}{\partial z} z, \tag{3.39}$$

where $\pm\mu$ is the magnetic moment of the atom for the $M = \pm 1$ states, respectively. We expand the force about $z = 0$, analogous to the Doppler shift case, where we expand about $v = 0$,

$$\langle \mathbf{F} \rangle = \sum_\pm \frac{\partial \langle \mathbf{F}_\pm \rangle}{\partial z} z = \sum_\pm \frac{\partial \langle \mathbf{F}_\pm \rangle}{\partial \Delta_0^\pm} \frac{\partial \Delta_0^\pm}{\partial z} z \equiv -K z\, \hat{\mathbf{e}}_z, \tag{3.40}$$

where K is the effective spring constant. Now, this result is identical in form to that leading to Eq. 3.33 for the damping force, where the net force arises

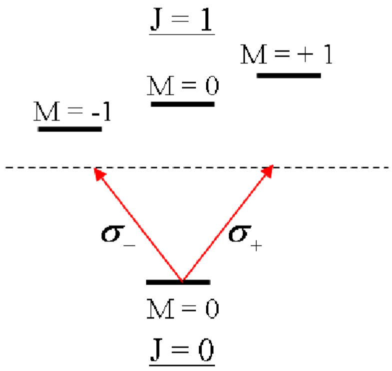

Figure 3.7: Optical forces on a three-level atom. Absorption of a photon from the right and left circularly polarized ($\sigma_\pm$) beams produce transitions from the $J = 0, M = 0$ ground state to the $J = 1, M = \pm 1$ states, respectively, to conserve angular momentum. For an atom with $z > 0$, where the magnetic field has $B_z > 0$, the $M = 1$ state is Zeeman tuned further from resonance, while the $M = -1$ state is tuned closer to resonance, producing a net restoring force along $-z$.

from the velocity dependent Doppler detuning $\mp kv$ for the opposing beams. Hence, to find the spring constant for the MOT, we simply replace kv by the Zeeman detuning, so that $-2kv \to -2\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}z$ in Eq. 3.33. Then, with the same notation as in Eq. 3.33,

$$\begin{aligned}\langle \mathbf{F} \rangle &= \frac{-2\Delta_0/\gamma_{21}^2}{\left[1+\frac{\Delta_0^2}{\gamma_{21}^2}+I/I_{sat}\right]^2}\frac{I}{I_{sat}}(-2\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}z)\,\frac{\gamma_s}{2}\hbar k\hat{\mathbf{e}}_z \\ &= \hbar k\,\hat{\mathbf{e}}_z\frac{\delta I'}{(1+\delta^2+I')^2}\,4\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}z = -Kz\,\hat{\mathbf{e}}_z. \qquad (3.41)\end{aligned}$$

Hence, Eq. 3.41 shows that the spring constant for the MOT in the z-direction is given by,

$$K = 4k\mu\,\frac{\partial B_z}{\partial z}\,\frac{-\delta I'}{(1+\delta^2+I')^2}. \qquad (3.42)$$

Note that the spring constant is positive for negative detuning and has the same structure as Eq. 3.35. Hence, the maximum spring constant arises for $\delta_{opt} = -1$ and $I' = 2$, as for maximum damping. Then, with $1{\cdot}2/(1{+}1{+}2)^2 =$

1/8, we obtain

$$\boxed{K_{opt} = \frac{1}{2} k\mu \frac{\partial B_z}{\partial z}} \tag{3.43}$$

for $I = 2I_{sat}$ and $\omega - \omega_0 = -\gamma_s/2$, where the magnetic field gradient is evaluated at $z = 0$. Note that the magnetic moment μ causes a frequency shift of $\mu/\hbar = 2\pi \times 1.4$ MHz/G for one Bohr magneton.

Trap Depth

The effective force of the MOT beams for one direction (z) is

$$F_z = -K\, z, \tag{3.44}$$

so that the effective trapping potential is harmonic $U = Kz^2/2$. The maximum trap depth is determined by the largest z for which the atoms still interact with the light, Fig. 3.8.

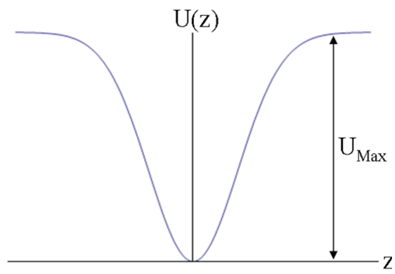

Figure 3.8: The maximum depth of a magneto-optical trap is determined by the largest detuning such that the light still interacts with the atoms, i.e., the largest z, where the Zeeman shift is approximately equal to the linewidth.

In this case, the maximum Zeeman shift must be less than or equal to the linewidth. We take

$$\left|\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}\right| z_{Max} = 2\gamma_{21} = \gamma_s. \tag{3.45}$$

Then, the magnitude of z_{Max} is

$$z_{Max}(cm) = \frac{\gamma_s(rad/s)}{\left|\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}\right|} = \frac{\gamma_s(Hz)}{\left|\frac{\mu}{h}\frac{\partial B_z}{\partial z}\right|(Hz/cm)} \tag{3.46}$$

For a typical atom, μ is approximately a Bohr magneton, and $\mu/h = 1.4$ MHz/G as noted above. Using $\partial B_z/\partial z = 30$ G/cm, and $\gamma_s(Hz) = 1/(2\pi\tau_s) = 5.9$ MHz for ^{6}Li, we have $z_{Max} = 5.9/(1.4)(30) = 0.14$ cm $= 1.4$ mm, which is the nominal size of a typical MOT. Here, we have neglected the effective repulsive force arising from absorption of photons from the MOT beams, which becomes important when the number of atoms is large. This leads to spontaneous photons, which are reabsorbed, producing an outward repulsive force that we will treat in the homework problems. Neglecting this effect, the corresponding maximum trap depth is

$$\begin{aligned} U_{Max} &= \frac{1}{2}K_{opt}z_{Max}^2 \\ &= \frac{1}{2}\frac{1}{2}k\mu\frac{\partial B_z}{\partial z}\left(\frac{\gamma_s}{\left|\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}\right|}\right)^2 = \gamma_s\frac{\hbar k}{4}\left(\left|\frac{\mu}{\hbar}\frac{\partial B_z}{\partial z}\right|\right)^{-1}\gamma_s \\ &= \frac{\hbar\gamma_s}{4}kz_{Max} = k_B T_{Doppler}\,\frac{kz_{Max}}{2}. \end{aligned} \tag{3.47}$$

Eq. 3.47 shows that the maximum MOT depth is *inversely* proportional to the magnetic field gradient.

Now, $kz_{Max}/2 = \pi z_{Max}/\lambda$. For ^{6}Li, where $\lambda = 0.67\,\mu$m, we have $kz_{Max}/2 = 3.14(0.14)/(0.67\times 10^{-4}) = 6.6\times 10^3$. Using $T_{Doppler} = 0.14$ mK, as shown above, we find $U_{Max}/k_B = 0.9\,K >> T_{Doppler}$. Typical MOT depths are the order of one Kelvin, large compared to the energy scales of cold atoms, which typically are well below a milli-Kelvin.

Capture Velocity

Optical molasses cools the atoms into the MOT, but the cooling only works if the velocity is low enough that the Doppler shift is at most $kv = \gamma_s$. The capture velocity is then

$$kv_c = \gamma_s = \frac{1}{\tau_s}$$

or

$$v_c(cm/s) = \lambda(cm)\,\gamma_s(Hz). \tag{3.48}$$

For ^{6}Li, with the above parameters, the capture velocity $v_c = 400$ cm/s or 4 m/s. This can be increased by using larger detunings $\omega - \omega_0$ from resonance and higher intensity I/I_{sat} during the MOT loading phase. Following the loading, optimum cooling can be obtained by lowering the MOT beam intensities and tuning closer to resonance.