PHYS 690 Quantum and Nonlinear Optics

Problem set # 5 *(discussion date - December 8)*

P1 Hardy's paradox
Explain the setup for Hardy's thought experiment (there are many descriptions available on-line, including in wikipedia) and what is "paradoxical" about it.

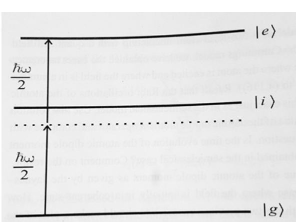
P2 Bogoliubov transformation
Many nonlinear processes in quantum optics can be described using Bogoliubov transformations between two input ($\hat{a}_{1,2}$ and two output $\hat{b}_{1,2}$ modes of optical fields:
$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix}$
What relations should the coefficients $A_{i,j}$ obey to preserve the commutation relationships between the output operators $\hat{b}_{1,2}$? Use them to show that photon number difference between the two channels is conserved, i.e., $\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2 = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2$

P3 Parametric down conversion
Using the PDC Hamiltonian $\hat{H} = \mu \hat{a}_s^\dagger \hat{a}_i^\dagger + \mu^* \hat{a}_s \hat{a}_i$, find equations describing time evolution of the signal $\hat{a}_s$ and idler $\hat{a}_i$ fields. Assuming short interaction time, write the Bogoliubov transformation matrix.

P4 The output of a parametric optical amplifier, seeded with a coherent vacuum signal, is described by the following wave function:
$|\psi\rangle = \frac{1}{\cosh\xi} \sum_{n=0}^{\infty} (\tanh\xi)^n |n,n\rangle,$
where ξ is the parameter describing the nonlinearity of the amplifier medium. Show that in case of the weak interaction $\xi \ll 1$ such state predominantly lead to the generation of single pairs, namely that:
$|\psi\rangle \simeq |0,0\rangle + \xi|1,1\rangle.$
This system can be used as a source of heralded single photons, with a detection of, e.g., an idler photon announcing the presence of the signal photon in the other channel. If the detector used for heralding is a "click-no click" detector and won't be able to distinguish between one or two photons, what is the probability of a two signal photon state production?

P5 Quantum Rabi oscillations.
Assume that an atom initially is in the excited state, the electromagnetic field is in a number state $|n\rangle$, and its frequency matches the transition frequency of a two-level system (resonant conditions). Using fully quantized description of the system, derive the quantum state of the system as a function of time.

P7 Two-photon Jaynes-Cummings model



A resonant two-photon extension of the Jayne-Cummongs model is described by the effective Hamiltonian $\hat{H}_{eff} = \hbar\eta\left(\hat{a}^2\hat{\sigma}_+ + (\hat{a}^\dagger)^2\hat{\sigma}_-\right)$. This Hamiltonian represents two-photon absorption and emission between atomic levels $|g\rangle$ and $|e\rangle$ of same parity, and their energy difference $\hbar\omega$ matches exactly the twice frequency of the optical field. The figure shows the schematics of the process, where the broken line represents a virtual intermediate state of the opposite parity (can be disregarded in the calculations). Obtain the dressed states for this system. Assuming that the atoms is initially in the ground state and the optical field is in a number state $|n\rangle$, calculate the time-dependent atomic inversion.

P7 Light-atom entanglement
In class we focused on resonant Jaynes-Cummings model, but it is possible to obtain the analytic expression for the quantum state evolution for a far-detuned laser field ($\omega - \omega_0 = \Delta$. In particular, if the initial state of the system is an atom in the superposition of the ground and excited states $(|a\rangle + |b\rangle)/\sqrt{2}$, and the optical field is in a coherent state $|\alpha\rangle$, the time-dependent wave function is described by the following expression:
$|\psi\rangle = \left(|\alpha e^{i\chi t}\rangle|a\rangle + e^{-i\chi t}|\alpha e^{-i\chi t}\rangle|b\rangle\right)/\sqrt{2},$
where $\chi = g^2/\Delta$. If $\chi t = \pi/2$ we can obtained the following entangled state between the light and the atom:

$|\psi\rangle = (|i\alpha\rangle|a\rangle - i| - i\alpha\rangle|b\rangle)/\sqrt{2}$.
Suppose that there is some way to determine the state of the atom. What would be the state of the optical field if the atom is detected in state $|b\rangle$ or state $|a\rangle$? What if it is possible to detect the atom in the superposition states $(|a\rangle \pm |b\rangle)/\sqrt{2}$, what optical field states are generated in this case?

P8 Black body radiation law
Derive the dependence of the average number $\bar{n}$ of photons in equilibrium with a two-level atom, using the relationship for emission and absorption probabilities we derived in class:
$\frac{P_{ems}}{P_{abs}} = \frac{\bar{n}+1}{\bar{n}}$,
and the Boltzmann law for the atomic populations:
$N_b/N_a = exp\left(-\frac{\hbar\omega}{k_B T}\right)$.