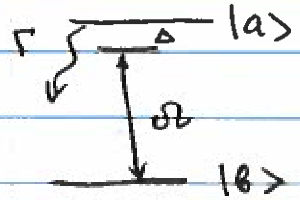
① Two-level system



$$\dot\rho_{aa} = -\Gamma\rho_{aa} + i\Omega^*\rho_{ba} - i\Omega\rho_{ab}$$
$$\dot\rho_{bb} = -\dot\rho_{aa} = \Gamma\rho_{aa} - i\Omega^*\rho_{ba} + i\Omega\rho_{ab}$$

$$\dot\rho_{ab} = -\left(\frac{\Gamma}{2} - i\Delta\right)\rho_{ab} + i\Omega^*(\rho_{bb}-\rho_{aa})$$

a) Steady state: $\frac{\partial}{\partial t}\rho_{ij} = 0$

$$\rho_{ab} = \frac{i\Omega^*(\rho_{bb}-\rho_{aa})}{\Gamma/2 - i\Delta}$$

$$-\Gamma\rho_{aa} + \frac{|\Omega|^2(\rho_{bb}-\rho_{aa})}{\Gamma/2 - i\Delta} + \frac{|\Omega|^2(\rho_{bb}-\rho_{aa})}{\Gamma/2 + i\Delta} = 0$$

$(\rho_{bb} = 1-\rho_{aa})$

$$-\Gamma\rho_{aa} + |\Omega|^2(1-2\rho_{aa})\frac{\Gamma}{\Gamma^2/4+\Delta^2} = 0$$

$$\rho_{aa} = \frac{|\Omega|^2}{\Gamma^2/4+\Delta^2+2|\Omega|^2}$$

$$\rho_{bb}-\rho_{aa} = 1-2\rho_{aa} = \frac{\Gamma^2/4+\Delta^2}{\Gamma^2/4+\Delta^2+2|\Omega|^2} > 0$$

for any parameters

no population inversion

b) The analytical solution exist for $\Delta = 0$

Then

$$\dot{\rho}_{ab} = -\frac{\Gamma}{2}\dot{\rho}_{ab} + i\Omega^*(\rho_{bb}-\rho_{aa})$$

$$\Omega^*\dot{\rho}_{ba} - \Omega\dot{\rho}_{ab} = -\frac{\Gamma}{2}(\Omega^*\rho_{ba} - \Omega\rho_{ab}) - 2i|\Omega|^2(\rho_{bb}-\rho_{aa})$$

$$\dot{\rho}_{bb} - \dot{\rho}_{aa} = 2\Gamma\rho_{aa} - 2i\Omega^*\rho_{ba} + 2i\Omega\rho_{ab}$$

$$\ddot{\rho}_{bb} - \ddot{\rho}_{aa} = 2\Gamma\dot{\rho}_{aa} - i2[\Omega^*\dot{\rho}_{ba} - \Omega\dot{\rho}_{ab}] =$$

$$= 2\Gamma\dot{\rho}_{aa} + \underbrace{i\frac{\Gamma}{\cancel{2}}(\Omega^*\rho_{ba} - \Omega\rho_{ab}]}_{\frac{\Gamma}{\cancel{2}}(\dot{\rho}_{aa} + \Gamma\rho_{aa})} - \cancel{2}4|\Omega|^2(\rho_{bb}-\rho_{aa})$$

$$= 3\Gamma\dot{\rho}_{aa} + \Gamma^2\rho_{aa} - \cancel{2}4|\Omega|^2(\rho_{bb}-\rho_{aa})$$

$$\rho_{aa} + \rho_{bb} = 1 \Rightarrow \rho_{aa} = \frac{1}{2}(1 - (\rho_{bb}-\rho_{aa}))$$

$$\Delta\rho = \rho_{bb} - \rho_{aa}$$

$$\Delta\ddot{\rho} = \frac{3\Gamma}{2}(\cancel{1} - \Delta\dot{\rho}) + \frac{\Gamma^2}{2}(1-\Delta\rho) - \cancel{2}4|\Omega|^2\Delta\rho = 0$$

$$\Delta\ddot{\rho} + \frac{3\Gamma}{2}\Delta\dot{\rho} + \left(\frac{\Gamma^2}{2} + \cancel{2}4|\Omega|^2\right)\Delta\rho - \frac{\Gamma^2}{2} = 0$$

$$\Delta\rho(t) = \underset{\text{homogeneous}}{\Delta\rho(t)} + \underset{\text{steady state}}{\Delta\rho_{st.st}}$$

$$\Delta\rho_{st.st} = +\frac{\Gamma^2}{\Gamma^2 + 8|\Omega|^2}$$

Homogeneous: $\Delta\ddot{\rho} + \frac{3\Gamma}{2}\Delta\dot{\rho} + \left(\frac{\Gamma^2}{2} + 4|\Omega|^2\right)\Delta\rho = 0$

$$\Delta\rho = e^{\lambda t}$$

$$\lambda^2 + \frac{3\Gamma}{2}\lambda + \left(\frac{\Gamma^2}{2} + 4|\Omega|^2\right) = 0$$

$$\lambda_{1,2} = -\frac{3\Gamma}{4} \pm \sqrt{\frac{9\Gamma^2}{16} - \frac{\Gamma^2}{2} - 4|\Omega|^2}$$

$$\lambda_{1,2} = -\frac{3\Gamma}{4} \pm i\sqrt{4|\Omega|^2 - \frac{\Gamma^2}{16}}$$

$4|\Omega|^2 < \Gamma^2/16$ — purely decaying solution

$4|\Omega|^2 > \Gamma^2/16$ — oscillating solution w/ decaying amplitude

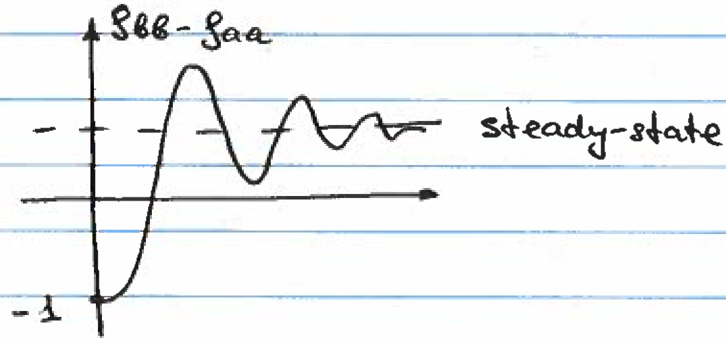
Initial condition $\Delta\rho = \rho_{bb} - \rho_{aa} = -1$

$$\Delta\rho(t) = A e^{-3\Gamma t/4} \cos\sqrt{4|\Omega|^2 - \Gamma^2/16}\, t + \frac{\Gamma^2}{\Gamma^2 + 8|\Omega|^2}$$

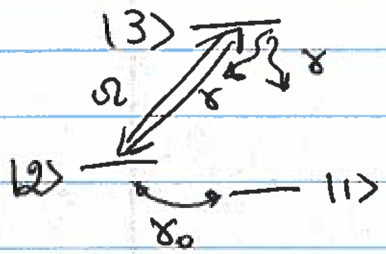
$t=0$

$$A + \frac{\Gamma^2}{\Gamma^2 + 8|\Omega|^2} = -1 \quad \Rightarrow \quad A = -\frac{2\Gamma^2 + 8|\Omega|^2}{\Gamma^2 + 8|\Omega|^2}$$

$$\Delta\rho(t) = \frac{\Gamma^2}{\Gamma^2 + 8|\Omega|^2}\left(1 - \frac{\Gamma^2 + 4|\Omega|^2}{\Gamma^2} e^{-3\Gamma t/4} \cos\sqrt{4|\Omega|^2 - \frac{\Gamma^2}{16}} \times t\right)$$



3. Optical pumping



$$\begin{cases} \dfrac{\partial \rho_{11}}{\partial t} = \gamma \rho_{33} - \gamma_0 (\rho_{11} - \rho_{22}) = 0 \\ \dfrac{\partial \rho_{22}}{\partial t} = \gamma \rho_{33} + \gamma_0 (\rho_{11} - \rho_{22}) - i\Omega^* \rho_{23} + i\Omega \rho_{32} = 0 \\ \dfrac{\partial \rho_{32}}{\partial t} = -(\gamma - i\Delta)\rho_{32} + i\Omega(\rho_{22} - \rho_{33}) \\ \rho_{33} + \rho_{11} + \rho_{22} = 1 \end{cases}$$

$$2\gamma \rho_{33} = i\Omega^* \rho_{23} - i\Omega \rho_{32} = |\Omega|^2 (\rho_{22} - \rho_{33}) \left(\frac{1}{\gamma + i\Delta} + \frac{1}{\gamma - i\Delta} \right)$$

$$\rho_{33} = \frac{|\Omega|^2}{\gamma^2 + \Delta^2} (\rho_{22} - \rho_{33})$$

$$\rho_{33} = \frac{|\Omega|^2}{\gamma^2 + \Delta^2 + |\Omega|^2} \rho_{22}$$

$$\gamma \rho_{33} = \gamma_0 (\rho_{11} - \rho_{22}) = \gamma_0 (1 - \rho_{33} - 2\rho_{22})$$

$$\rho_{33} (\gamma_0 + \gamma) + 2\gamma_0 \rho_{22} = \gamma_0$$

$$\rho_{33} = \frac{\gamma_0}{\gamma + 3\gamma_0 + 2\gamma_0 \dfrac{\gamma^2 + \Delta^2}{|\Omega|^2}} \xrightarrow[|\Omega| \to \infty]{\gamma_0/\gamma \ll 1} 0$$

$$\rho_{22} = \frac{\gamma^2 + \Delta^2 + |\Omega|^2}{|\Omega|^2} \; \frac{\gamma_0}{\gamma + 3\gamma_0 + 2\gamma_0 \dfrac{\gamma^2 + \Delta^2}{|\Omega|^2}} \xrightarrow[|\Omega| \to \infty]{\gamma_0/\gamma \ll 1} 0$$

$$\rho_{11} = 1 - \rho_{22} - \rho_{33} = \frac{|\Omega|^2 + \dfrac{\gamma_0}{\gamma}(|\Omega|^2 + \gamma^2 + \Delta^2)}{|\Omega|^2 \left(1 + \dfrac{3\gamma_0}{\gamma}\right) + \dfrac{2\gamma_0}{\gamma}(\gamma^2 + \Delta^2)} \xrightarrow[\gamma_0/\gamma \ll 1]{|\Omega| \to \infty} 1$$

4. $$\chi_p(\delta) = \frac{i p_{13}^2}{\hbar \varepsilon_0} N \quad \frac{\Gamma_{12}}{\Gamma_{12}\Gamma_{13} + |\Omega|^2}$$

$$\Gamma_{12} = \underbrace{\gamma_{12}}_{=\gamma_0} - i\delta \qquad \Gamma_{13} = \underbrace{\gamma_{13}}_{=\gamma} - i\delta \qquad \Delta_2 = 0$$

$$\chi_p(\delta) = \frac{i p_{13}^2}{\hbar \varepsilon_0} N \quad \frac{\gamma_0 - i\delta}{(\gamma_0\gamma - \delta^2 + |\Omega|^2) - i\delta(\gamma + \gamma_0)} =$$

$$= \frac{i p_{13}^2}{\hbar \varepsilon_0} N \left[\frac{(\gamma_0(\gamma_0\gamma + |\Omega|^2) + \delta^2\gamma) - i(\delta[|\Omega|^2 - \gamma_0^2 - \delta^2])}{(\gamma_0\gamma - \delta^2 + |\Omega|^2)^2 + \delta^2(\gamma_0 + \gamma)^2} \right]$$

$$n = 1 + \frac{Re(\chi)}{2} = 1 + \frac{p_{13}^2 N}{\hbar \varepsilon_0} \quad \frac{\delta \cdot (|\Omega|^2 - \gamma_0^2 - \delta^2)}{(\gamma_0\gamma - \delta^2 + |\Omega|^2)^2 + \delta^2(\gamma_0 + \gamma)^2}$$

a) $$\frac{\partial n}{\partial \omega} = \frac{\partial n}{\partial \delta}\bigg|_{\delta=0} = \frac{p_{13}^2 N}{\hbar \varepsilon_0} \quad \frac{|\Omega|^2 - \gamma_0^2}{\cancel{(\gamma_0\gamma + |\Omega|^2)}\,(\gamma_0\gamma + |\Omega|^2)^2} \approx \cancel{\frac{p_{13}^2 N}{\hbar \varepsilon_0} \frac{|\Omega|^2}{(\gamma_0\gamma)^2}}$$

$$\approx \frac{p_{13}^2 N}{\hbar \varepsilon_0} \quad \frac{|\Omega|^2}{(\gamma_0\gamma + |\Omega|^2)^2}$$

$$v_g = \frac{c}{n(\delta=0) + \omega_{probe} \frac{\partial n}{\partial \delta}\Big|_{\delta=0}} = \frac{c}{1 + \frac{p_{13}^2 N}{\hbar \varepsilon_0} \nu_1 \frac{|\Omega|^2}{(\gamma_0\gamma + |\Omega|^2)^2}}$$

b) $$n(\delta) = 1 + \frac{\wp_{13}^2 N}{\hbar\varepsilon_0}\,\frac{\delta(|\Omega|^2 - \delta^2)}{(\gamma_0\gamma - \delta^2 + |\Omega|^2)^2 + \delta^2\gamma^2}$$ $\gamma_0 \ll \gamma, |\Omega|$

Calculating the maximum of this function at random ~~[illegible]~~ values of parameters is tough, so we will assume that $|\Omega|$ is fairly strong, such that $|\Omega| \gg \delta$

Then

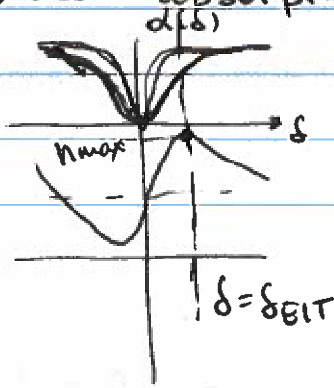
$$n(\delta) = 1 + \frac{\wp_{13}^2 N}{\hbar\varepsilon_0}\,\frac{\delta|\Omega|^2}{(\gamma_0\gamma + |\Omega|^2)^2 + \delta^2\gamma^2} = \frac{\wp_{13}^2 N}{\hbar\varepsilon_0}\,\frac{|\Omega|^2}{\gamma^2}\,\frac{\delta}{\gamma_{EIT}^2 + \delta^2}$$

where $\gamma_{EIT} = \gamma_0 + \frac{|\Omega|^2}{\gamma}$

In this case max. refractive index is at $\delta = \gamma_{EIT}$

$$n_{max} = 1 + \frac{\wp_{13}^2 N}{\hbar\varepsilon_0}\,\frac{|\Omega|^2}{2\gamma^2\gamma_{EIT}}$$

Even though EIT absorption vanishes @ $\delta = 0$ for $\gamma_{12} = 0$, n_{max} happens at $\delta = \gamma_{EIT}$, which is half-way off EIT peak center, so there will be ~~[illegible]~~ ~~[illegible]~~ some absorption for sure



5. From the class-notes, the susceptibility for the far-detuned case

$$\chi_p = i\alpha_0 \frac{\gamma_{13}}{\gamma_{13}-i\Delta_1} + i\alpha_0 \frac{|\Omega_2|^2}{\Delta_1^2} \frac{1}{\gamma_R + i(\delta-\delta_R)}$$

Two absorption peaks $\Delta_1=0$ (regular abs)

$\delta=\delta_R$ – two-photon absorption

$\delta=\Delta_1-\Delta_2-\omega_{12}$, so for $\delta=\delta_R$ $\Delta_1 \simeq \Delta_2+\omega_{12}$

At such detuning the contribution from the first term is small ($\Delta_2 \gg \gamma_{13}$), and can be potentially neglected

$$n = 1+\frac{Re[\chi]}{2} = 1-\alpha_0\underbrace{\frac{\gamma_{13}\Delta_1}{\gamma_{13}^2+\Delta_1^2}}_{\sim \gamma_{13}/\Delta_1 \ll 1} - \alpha_0 \frac{|\Omega_2|^2}{\Delta_1^2}\frac{(\delta-\delta_R)}{\gamma_R^2+(\delta-\delta_R)^2}$$

$$\left.\frac{\partial n}{\partial\omega}\right|_{\delta=\delta_R} \approx -\alpha \frac{|\Omega_2|^2}{\cancel{4}\Delta_1^2\gamma_R^2} = -\alpha \frac{|\Omega_2|^2/\Delta_1^2}{(\gamma_{12}+\gamma_{13}|\Omega_2|^2/\Delta_1^2)^2}$$

$$v_g = \frac{c}{n|_{\delta=\delta_R}+\nu_1 \left.\frac{\partial n}{\partial\omega}\right|_{\delta=\delta_R}} \approx \frac{c}{1-\alpha\nu_1\frac{|\Omega_2|^2/\Delta_1^2}{(\gamma_{12}+\gamma_{13}|\Omega_2|^2/\Delta_1^2)^2}} > c \text{ or } < 0$$

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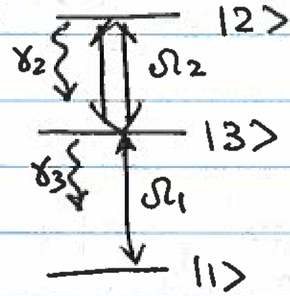

$$\hat{H} = \begin{pmatrix} \mathcal{E}_1 & 0 & \wp_{13}E_1 \\ 0 & \mathcal{E}_2 & \wp_{23}E_2 \\ \wp_{13}E_1 & \wp_{23}E_2 & \mathcal{E}_3 \end{pmatrix}$$

$\mathcal{E}_i$ – energies of the states
E_i – total electric fields

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$$

$$\begin{cases} \dot{\rho}_{12} = (\mathcal{E}_1 - \mathcal{E}_2)\rho_{12} + \rho_{23}\,\wp_{13}E_1 - \rho_{13}\,\wp_{23}E_2 \\ \dot{\rho}_{13} = (\mathcal{E}_1 - \mathcal{E}_3)\rho_{13} + (\rho_{11} - \rho_{33})\,\wp_{13}E_1 \\ \dot{\rho}_{32} = (\mathcal{E}_3 - \mathcal{E}_2)\rho_{32} + (\rho_{33} - \rho_{22})\,\wp_{23}E_2 \\ \quad + \text{ eqns for populations} \end{cases}$$

RWA
$$\rho_{12} = \tilde{\rho}_{12}\, e^{-(\nu_1+\nu_2)t}$$
$$\rho_{13} = \tilde{\rho}_{13}\, e^{-\nu_1 t}$$
$$\rho_{32} = \tilde{\rho}_{32}\, e^{-\nu_2 t}$$

As before, we keep only slowly varying terms in the electric fields

$$E_{1,2} = \frac{1}{2}\mathcal{E}_{1,2}\, e^{-i\nu_{1,2}t} + \frac{1}{2}\mathcal{E}^*_{1,2}\, e^{i\nu_{1,2}t}$$

$$\Omega_{1,2} = \frac{\wp_{2,1/3}\,\mathcal{E}_{1,2}}{2\hbar}$$

$$\dot{\rho}_{11} = \gamma_3 \rho_{33} - i\Omega_1 \rho_{13} + i\Omega_1^* \rho_{31}$$

$\rho_{33} = 1 - \rho_{11} - \rho_{22}$

$$\dot{\rho}_{22} = -\gamma_2 \rho_{22} - i\Omega_2 \rho_{23} + i\Omega_2^* \rho_{32}$$

$$\dot{\rho}_{21} = -\Gamma_{21} \rho_{21} + i\Omega_2^* \rho_{31} - i\Omega_1 \rho_{23}$$

$$\dot{\rho}_{31} = -\Gamma_{31} \rho_{31} + i\Omega_2 \rho_{21} + i\Omega_1 (\rho_{11} - \rho_{33})$$

$$\dot{\rho}_{32} = -\Gamma_{32} \rho_{32} + i\Omega_1 \rho_{12} + i\Omega_2 (\rho_{22} - \rho_{33})$$

The equations are similar to Λ scheme, but decays are different

$$\Gamma_{21} = \frac{\gamma_2}{2} - i\delta$$

$$\Gamma_{31} = \frac{\gamma_3}{2} - i\Delta_1$$

$$\Gamma_{32} = \frac{\gamma_2 + \gamma_3}{2} - i\Delta_2$$

$\Delta_1 = \nu_1 - \omega_{13}$

$\Delta_2 = \nu_2 - \omega_{32}$

$\delta = \Delta_1 + \Delta_2 = (\nu_1 + \nu_2) - (\omega_{13} + \omega_{32})$

We can make similar assumptions about density ~~matrices~~ element, assuming that Ω_1 is small

$\rho_{22} = \rho_{33} = 0 \qquad \rho_{23} = 0$ (b/w empty levels)

$$\begin{cases} \dot{\rho}_{21} = -\Gamma_{21} \rho_{21} + i\Omega_2 \rho_{31} = 0 \\ \dot{\rho}_{31} = -\Gamma_{31} \rho_{31} + i\Omega_2 \rho_{21} + i\Omega_1 \end{cases}$$

$$\rho_{31} = i\Omega_1 \frac{\Gamma_{12}}{\Gamma_{12}\Gamma_{13} + |\Omega_2|^2}$$

$$\chi = \frac{\wp_{13}^2}{\hbar\varepsilon_0} \frac{\rho_{31}}{\Omega_1} = i \frac{\wp_{13}^2}{\hbar\varepsilon_0} \frac{\Gamma_{12}}{\Gamma_{12}\Gamma_{13} + |\Omega_2|^2}$$

Best absorption supression $\delta = 0 \quad \Delta_1 = 0$

$\Gamma_{21} = \gamma_2/2 \qquad \Gamma_{31} = \gamma_3/2$

$$\alpha = \frac{k}{2} \frac{\wp_{13}^2}{\hbar \varepsilon_0} \frac{2\gamma_2}{\gamma_2\gamma_3 + 4|\Omega_2|^2}$$

EIT conditions $\quad |\Omega_2|^2 \gg \frac{\gamma_2\gamma_3}{4}$