Physics 622, Midterm test
March 4, 2016

Name:______________________________

Problem 1 (30 points)
A particle of charge q and mass m is bound in the ground state of a one-dimensional harmonic oscillator potential with frequency ω_0. Consider a perturbation in the form of a weak time-dependent spatially uniform electric field $E(t) = E_0\Theta(t)\cos(\omega t)e^{-\frac{t}{\tau}}$. Calculate the probability of finding the system in an excited state n at time $t \gg \tau$, up to the first order. You may assume that for any two harmonic oscillator states n' and n

$$\langle n'|\hat{x}|n\rangle = \sqrt{\hbar/2m\omega_0}\left(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}\right).$$

Problem 2 (35 points)
Consider an isotropic harmonic oscillator of mass m in two dimensions, such that its Hamiltonian is

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2).$$

A weak perturbation $\hat{V} = \alpha m\omega^2 xy$, where α is a dimensionless real number much smaller than one.
Find the shifts of the ground and first excited energy states.

Problem 3 (15 points)
Determine the hyperfine structure of the $5S_{1/2}$ level of ^{87}Rb due to the interaction of electron spin S and nuclear spin $I = 3/2$ described by $\hat{H}_{hf} = A\,\vec{I}\cdot\vec{S}$. Make and argument why $\vec{F} = \vec{I} + \vec{S}$ is a good quantum number. What are the degeneracies of each energy level?

Problem 4 (20 points)
A particle of mass m is in the bound state of a δ-potential well $V(x) = -\alpha\delta(x)$. Suddenly the value of the "depth" parameter, α, doubles. Find the probability that the particle will still be bound after the change.
***Bonus** (+10 points)* Find the momentum distribution for the particle that has been "kicked out" of the original well.

Midterm solutions 3/4/16

1. $\hat{H}_0|n\rangle = \hbar\omega_0(n+\frac{1}{2})\,|n\rangle$

$$\hat{V} = -q\vec{r}\vec{E} = -q x E_0\,\theta(t)\cos\omega t\; e^{-t/\tau}$$

Initial state – ground $|i\rangle = |0\rangle$

$$c_n(t) = \frac{i}{\hbar}\int_0^t \langle 0|\hat{V}|n\rangle e^{i\omega_{n0}t'}dt' =$$

$$= \frac{i}{\hbar}(-qE_0)\int_0^t \langle 0|\hat{x}|n\rangle \cos\omega t'\, e^{-t'/\tau}\, e^{in\omega_0 t'}dt'$$

$$\langle 0|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\;\delta_{n,1}$$

$$c_1(t) = -i\frac{qE_0}{\sqrt{2m\hbar\omega_0}}\,\frac{1}{2}\int_0^t\left(e^{i(\omega+\omega_0)t'-t'/\tau} + e^{i(-\omega+\omega_0)t'-t'/\tau}\right)dt'$$

$$= i\frac{qE_0}{2\sqrt{2m\hbar\omega_0}}\left[\frac{1-e^{i(\omega+\omega_0)t'-t'/\tau}}{i(\omega+\omega_0)-1/\tau} + \frac{1-e^{i(-\omega+\omega_0)t'-t'/\tau}}{i(-\omega+\omega_0)-1/\tau}\right]_0^t$$

$$\xrightarrow{t\gg\tau} i\frac{qE_0}{2\sqrt{2m\hbar\omega_0}}\left[\frac{1}{i(\omega+\omega_0)-1/\tau} + \frac{1}{i(-\omega+\omega_0)-1/\tau}\right] =$$

$$= i\frac{qE_0}{\sqrt{2m\hbar\omega_0}}\,\frac{i\omega_0\tau-1}{(i\omega_0\tau-1)^2+\omega^2\tau^2} = i\frac{qE_0}{\sqrt{2m\hbar\omega_0}}\,\frac{i\omega_0\tau-1}{(\omega^2-\omega_0^2)\tau^2+1-2i\omega_0\tau}$$

$$P = |c_1|^2 = \frac{(qE_0)^2}{2m\hbar\omega_0}\,\frac{(\omega_0\tau)^2+1}{[(\omega-\omega_0)^2\tau^2+1]^2+4\omega_0^2\tau^2}$$

3. $\hat{H}_{HF} = A\hat{\vec{I}}\cdot\hat{\vec{S}} = \frac{1}{2}A\left[(\hat{\vec{I}}+\hat{\vec{S}})^2 - \hat{I}^2 - \hat{S}^2\right)$

We can use $|\ell s j i F m_F\rangle$ basis

$$\Delta_{HF} = \frac{\hbar^2}{2}A\left(F(F+1) - I(I+1) - S(S+1)\right) = \frac{\hbar^2}{2}A\left(F(F+1) - \frac{15}{4} - \frac{3}{4}\right)$$

$F = I+S = 2$ $\qquad \Delta_{HF} = \frac{3\hbar^2}{4}A$

$F = I-S = 1$ $\qquad \Delta_{HF} = -\frac{5\hbar^2}{4}A$

2.

Ground state $|00\rangle$ $E_{00} = \hbar\omega$

First excited state : $|01\rangle$ and $|10\rangle$

$E_{01} = E_{10} = 2\hbar\omega$

$\hat{V} = \alpha m\omega^2 xy$

Ground state: no first order shift

Only non-zero elements

$\langle 00|\hat{V}|10\rangle = \langle 00|\hat{V}|11\rangle = \hbar/2m\omega \cdot \alpha m\omega^2 = \frac{\alpha\hbar\omega}{2}$

$$\Delta E_0 = \frac{(\alpha\hbar\omega/2)^2}{\hbar\omega - 3\hbar\omega} = -\frac{\alpha^2\hbar\omega}{8}, \quad E_{00} = \hbar\omega - \alpha^2\frac{\hbar\omega}{8}$$

First excited state: double-degenerate

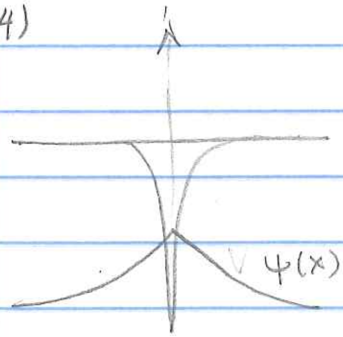
$V_{10} = \langle 01|\hat{V}|10\rangle = \langle 10|\hat{V}|01\rangle = \frac{\alpha\hbar\omega}{2}$

$$\det|\hat{V} - \lambda\hat{I}| = 0 \quad \begin{vmatrix} -\lambda & V_{10} \\ V_{10} & -\lambda \end{vmatrix} = 0$$

$\lambda^2 = V_{10}^2 \quad \lambda_{1,2} = \pm V_{12}$

$$E_{\pm} = 2\hbar\omega \pm \alpha\frac{\hbar\omega}{2}$$

4)



$$\psi(x) = A e^{-\varkappa|x|}$$

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}(\psi_+' - \psi_-')\Big|_{x=0} - \alpha\psi(0) = 0$$

$$\frac{\hbar^2}{2m}\cdot 2\varkappa A - \alpha A = 0$$

$$\varkappa = \frac{m\alpha}{\hbar^2}$$

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = 2\int_0^{\infty} A^2 e^{-2\varkappa x}\, dx$$

$$= A^2 \frac{1}{\varkappa} = 1 \qquad A = \sqrt{\varkappa}$$

For $t<0$ $\qquad \psi_0(x) = \sqrt{\varkappa}\, e^{-\varkappa|x|}$

For $t>0$ $\qquad \psi(x) = \sqrt{2\varkappa}\, e^{-2\varkappa|x|}$

$$\psi_0(x) = c\,\psi(x) + \sum_k \beta_k e^{ik|x|}$$

$$P_{stay} = |c|^2 = \left|\int_{-\infty}^{+\infty} \psi_0(x)\psi(x)\, dx\right|^2 = 2\varkappa^2 \left|2\int_0^{+\infty} e^{-3\varkappa x}\, dx\right|^2 =$$

$$= 2\varkappa^2 \left|\frac{2}{3\varkappa}\right|^2 = \underline{\frac{8}{9}}$$