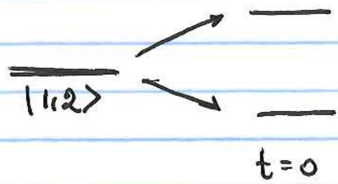
(Intro – post-previous class discussion)

Two-level system

Free precession vs Rabi oscillations




$|+\rangle = \frac{1}{2}(|1\rangle + |2\rangle)$

$|-\rangle = \frac{1}{2}(|1\rangle - |2\rangle)$

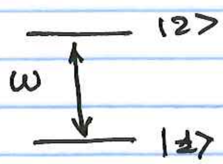
$|\psi_0\rangle = |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ at $t=0$

$$P_1(t) = |\langle 1|\psi(t)\rangle|^2 = \frac{1}{2}\left|\langle 1|\, |+\rangle e^{-iE_+t/\hbar} + |-\rangle e^{-iE_-t/\hbar}\right|^2 =$$

$$= \cos^2 \frac{(E_+ - E_-)t}{2\hbar} \quad \text{oscillations}$$

Frequency is determined by the energy splitting. The presence of the oscillations depends on the initial state.

Driven two-level system




$c_1, c_2 \sim e^{i\omega t}$

$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$

$\hat{V} = \gamma |1\rangle\langle 2| e^{i\omega t} + h.c.$

$\omega \approx \omega_{12}$

two frequencies in our system

$$P_2 = |c_2|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + \frac{(\omega-\omega_{12})^2}{4}} \sin^2\left[t\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{12})^2}{4}}\right]$$

Rabi oscillations happen at much <u>slower</u> frequency that $\omega_{12}, \omega$ given by the perturbation strength and detuning from the resonance

Dyson series

Reminder: the problem we are solving

$$\hat{H}|\psi(t)\rangle = (\hat{H}_0 + \hat{V}(t))|\psi(t)\rangle = i\hbar \frac{\partial \psi}{\partial t}$$

$$|\psi(t)\rangle = \sum_n c_n(t)\, e^{-\frac{iE_n t}{\hbar}} |n\rangle$$

$$i\hbar \frac{\partial c_n}{\partial t} = \sum_{n'} c_{n'} V_{nn'}(t)\, e^{i\omega_{nn'}t} \qquad \omega_{nn'} = \frac{E_n - E_{n'}}{\hbar}$$

This is exact solution, no assumptions about strength of perturbation

Explicitely considering the perturbation small

$$c_n(t) = c_n^{(0)} + \underbrace{\underset{\sim V}{c_n^{(1)}} + \underset{\sim V^2}{c_n^{(2)}} + \dots}_{\text{small corrections}}$$

For simplicity let's assume that at $t=0$ the system was in state $|i\rangle$

$$c_n(0) = \delta_{in}$$

Then rewriting $i\hbar \frac{dc_n^{(1)}}{dt} = \sum_{n'} c_{n'}^{(0)} V_{nn'}(t)\, e^{i\omega_{nn'}t}$

we get $i\hbar \frac{dc_n^{(1)}}{dt} = V_{ni}(t)\, e^{i\omega_{ni}t}$

and $c_n^{(1)} = -\frac{i}{\hbar}\int_{t_0}^{t} V_{ni}(t')\, e^{i\omega_{ni}t'}\, dt'$

For the second order

$$i\hbar \frac{dc_n^{(2)}}{dt} = \sum_{n'} c_{n'}^{(1)} V_{nn'}(t)\, e^{i\omega_{nn'}t}$$

$$c_n^{(2)} = -\frac{i}{\hbar}\sum_{n'}\int_{t_0}^{t} c_{n'}^{(1)}(t')\, V_{nn'}(t')\, e^{i\omega_{nn'}t'} =$$

$$= \left(-\frac{i}{\hbar}\right)^2 \sum_{n'} \int_{t_0}^{t} V_{nn'}(t')\, e^{i\omega_{nn'}t'} \int_{t_0}^{t'} V_{n'i}(t'')\, e^{i\omega_{n'i}t''}\, dt''$$

in principle, we can recreate the whole series to obtain exact solution → question of convergence.

Stick to small perturbation $c_n^{(1)} \ll 1$, $c_n^{(2)} \ll c_n^{(1)}$

often this limits the duration of perturbation.

$$c_n(t) = \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t} V_{ni}(t)\, e^{i\omega_{ni} t'}\, dt'$$

More general operator treatment

let's introduce notation $|\alpha, t_0; t\rangle$ as the state, initialized at $t = t_0$

Then the propagator operator $U_I(t, t_0)$ is define as

$$|\alpha, t_0; t\rangle = U_I(t, t_0)\,|\alpha, t_0; t_0\rangle$$

Here we operate in the interaction picture, where the eigenstate evolve in time according to the unperturbed hamiltonian:

$$|n, t_0; t\rangle = e^{-iE_n t/\hbar}\,|n, t_0; t_0\rangle$$

In this picture the perturbation is

$$\hat{V}_I(t) = e^{+i\hat{H}_0 t/\hbar}\, \hat{V}\, e^{-i\hat{H}_0 t/\hbar}$$

The remaining time dependence in the states is due to $\hat{V}(t)$

$$i\hbar \frac{\partial|\alpha, t_0; t\rangle}{\partial t} = \hat{V}_I\,|\alpha, t_0; t\rangle$$

and $$i\hbar \frac{d\, U_I(t, t_0)}{dt} = \hat{V}_I(t)\, \hat{U}_I(t, t_0) \quad \text{with } U_I(t_0, t_0) = 1$$

thus

$$U_I(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t')\, U_I(t', t_0)\, dt' =$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t')\left[1 - \frac{i}{\hbar}\int_{t_0}^{t''} V_I(t'')\, U_I(t'', t_0)\right) dt' =$$

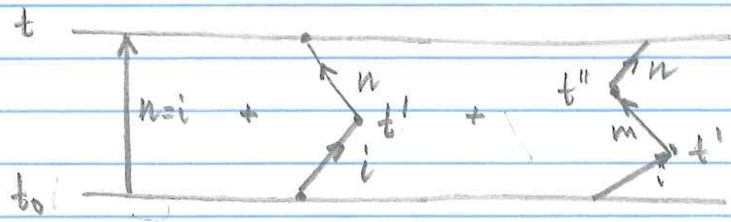
$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\, dt' + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} V_I(t') \int_{-\infty}^{t'} V_I(t'')\, U_I(t'', t_0)\, dt''$$

$$= 1 - \frac{i}{\hbar}\int_{-\infty}^{t} V_I(t')\,dt' + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t'} dt' \int_{t_0}^{t'} dt'' \; V_I(t')V_I(t'') -$$

$$- \frac{i}{\hbar}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \; V_I(t')V_I(t'')\,V_I(t''')$$

We can present this series using "Feynman diagrams"

$\langle n | U_I(t, t_0 | i \rangle =$




$= e^{-iE_n(t''-t')/\hbar}$

$\Rightarrow \langle m | V(t') | n \rangle$

If the probabilities of these various processe are comparable => strong interaction, need to consider all orders

$$= 1 - \frac{i}{\hbar}\int_{-\infty}^{t} V_I(t')\,dt' + \left(\frac{i}{\hbar}\right)\int_{-\infty}^{t'} V_I(t')\int_{-\infty}^{t''} V_I(t'')\,dt'\,dt'' - \left(\frac{i}{\hbar}\right)\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' \times V_I(t')V_I(t'')V_I(t''') + \dots$$

It is easy to see that the matrix elements of $U_I$ define the wave function expansion coefficient

$$\psi(t) = U_I\,\psi(t=0)$$

$$\sum_n c_n|n\rangle = U_I|i\rangle \Rightarrow c_n = \langle n|U_I|i\rangle$$

so if

$$U_I^{(1)}(t) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t)\,dt'$$

$$c_n^{(1)} = \langle n|U_I^{(1)}(t)|i\rangle = \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I|i\rangle\,dt' =$$

$$= \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t}\langle n|e^{+i\hat{H}_0t/\hbar}\,\hat{V}\,e^{-i\hat{H}_0t/\hbar}|i\rangle\,dt' =$$

$$= \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t} V_{ni}\,e^{i\omega_{ni}t}\,dt'$$

same!

## Transition probabilities

If in the beginning the system was in state $|i\rangle$, the probability to find it in some other state later $\equiv$ transition probability

$$P(i\to n) = |c_n(t)|^2 = |c_n^{(1)} + c_n^{(2)} + \dots|^2$$

First-order perturbation theory

$$c_n^{(1)} = -\frac{i}{\hbar}\int_{t_0}^{t} V_{ni}(t')\,e^{i\omega_{ni}t'}\,dt' \qquad n \neq i$$

$$P_{i\to n}(t) = |c_n^{(1)}|^2 \ll 1$$