Summary of main results up to date

$$\psi(\vec r) = \psi_0(\vec r) + f(\theta,\varphi)\,\frac{e^{ikr}}{r}$$

$\psi_0(\vec r) \downarrow\ \propto e^{ikz}$; $f(\theta,\varphi)$ or $f(\vec k, \vec k')$

$$f(\vec k,\vec k') = -\frac{2m}{4\pi\hbar^2}\int d^3\vec r'\, e^{-i\vec k'\vec r'}\, V(\vec r')\,\psi(\vec r')$$

Born approximation scattering = perturbation

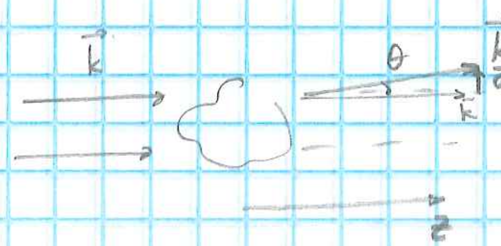
$$f(\vec k,\vec k') = -\frac{2m}{4\pi\hbar^2}\int V(\vec r')\, e^{i(\vec k-\vec k')\vec r'}\, d^3\vec r'$$

Born approximation is valid if $\frac{mV}{\hbar^2}R^2 \ll 1$

For fast particles $kR \gg 1$ $\left(\frac{mE}{\hbar^2}R^2 \gg 1\right)$

$e^{i(\vec k-\vec k')\vec r'}$ — fast-oscillating term, the integral averages to zero except for small angles $\vec k \approx \vec k'$

Fast particles — mostly forward scattering



$\vec k - \vec k' = \vec q \perp \vec k$

$q = k\sin\theta \approx k\theta$

Eikonal approximation ("fast" particle)

① Potential $V(x)$ changes very little at distances $\sim \lambda$ $(R \gg \lambda)$ $kR \gg 1$

② $E \gg V$ - fast particles.

Semiclassical approximation for the wave function

$$\psi^{(+)} \sim e^{iS(\vec{r})/\hbar}$$

$$-\frac{\hbar^2}{2m} \nabla^2 \left(e^{iS(\vec{r})/\hbar} \right) = \ \ = 0$$

$$= -\frac{\hbar^2}{2m} \left(\frac{(\nabla S)^2}{\hbar^2} + \cancel{\frac{\nabla^2 S}{\hbar}} \right) e^{iS(\vec{r})/\hbar}$$

Free particle

$S(\vec{r}) = \hbar \vec{k} \cdot \vec{r}$

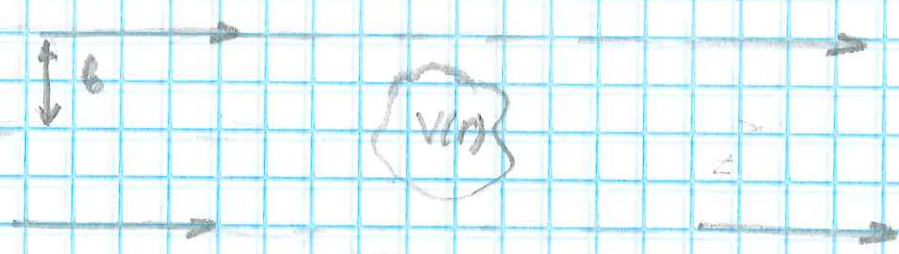
$\frac{1}{\hbar} \nabla S = \vec{k} + \vec{r} \, \nabla \vec{k}$

$\frac{1}{\hbar} \nabla^2 S = 2 \nabla \vec{k} + \nabla^2 k \cdot$

can neglect

Shrodinger equation $\equiv$ Hamilton-Jacobi eqn

$$-\frac{(\nabla S)^2}{2m} + V(r) = E = \frac{\hbar^2 k^2}{2m}$$



We assume that the particles deflection is small $\vec{k}' \approx \vec{k}$

$$f(\vec{k}', \vec{k}) = -\frac{2m}{4\pi\hbar^2} \int d^3 r' \, e^{-i\vec{k}'\vec{r}'} \, V(\vec{r}') \, \underline{\psi(\vec{r}')}$$

can keep only $\sim e^{ikz}$ term

straight motion

Cylindrical coordinates

$$r = \sqrt{b^2+z^2} \qquad \nabla = \frac{d}{dz}$$

$$\frac{1}{2m}\left(\frac{dS}{dz}\right)^2 + V(\sqrt{b^2+z^2}) = \frac{\hbar^2k^2}{2m}$$

$$S = \hbar\int_{-\infty}^{z}\sqrt{k^2 - \frac{2m}{\hbar^2}V(\sqrt{b^2+z'^2})}\,dz' \approx \{E>>V\}$$

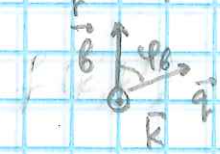
$$\approx \hbar\int_{-\infty}^{z}\left(k - \frac{m}{\hbar^2 k}V(\sqrt{b^2+z'^2})\right)dz' \approx \hbar k z - \frac{m}{\hbar k}\int_{-\infty}^{z}V(\sqrt{b^2+z'^2})\,dz'$$

$$\psi^+(\vec r) \approx \frac{1}{(2\pi)^{3/2}}\, e^{ikz}\cdot e^{-i\frac{m}{\hbar^2 k}\int_{-\infty}^{z}V(\sqrt{b^2+z'^2})\,dz'}$$

$$f(\vec k',\vec k) = -\frac{2m}{4\pi\hbar^2}\int d^3r'\, e^{-i\vec k'\vec r'}\, V(\vec r')\, e^{ikz'}\, e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z'}V(\sqrt{b^2+z''^2})\,dz''}$$

$$= -\frac{2m}{4\pi\hbar^2}\int d^3r'\, e^{i\vec q\vec r'}\, V(\sqrt{b^2+z'^2})\, e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z'}V(\sqrt{b^2+z''^2})\,dz''}$$

$$\vec q \perp \vec k, \vec k' \qquad \vec q\cdot\vec r' \approx \vec q\cdot\vec b \approx q\cdot b\cdot\cos\varphi_b = k\theta\cdot b\cos\varphi_b$$



$$f(\vec k,\vec k') = -\frac{2m}{4\pi\hbar^2}\int_0^{\infty} b\,db\int_{-\infty}^{+\infty}V(\sqrt{b^2+z'^2})\,e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z'}V(\sqrt{b^2+z''^2})\,dz''}\underbrace{\int_0^{2\pi}e^{ik\theta b\cos\varphi_b}\,d\varphi_b}_{2\pi J_0(kb\theta)}$$

$$= -\frac{m}{\hbar^2}\int_0^{\infty} b\,J_0(kb\theta)\,db\int_{-\infty}^{+\infty}V(\sqrt{b^2+z'^2})\,e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z'}V(\sqrt{b^2+z''^2})\,dz''}\,dz'$$

$$\frac{d}{dz'}\left(e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{z'}V(\sqrt{b^2+z''^2})\,dz''}\right) = e^{\cdots}\left(-\frac{im}{\hbar^2 k}\right)V(\sqrt{b^2+z'^2})$$

$$f(\vec k,\vec k') = \frac{i\hbar^2 k}{m}\cdot\frac{-m}{\hbar^2}\int_0 b\,J_0(kb\theta)\,e^{-\frac{im}{\hbar^2 k}\int_{-\infty}^{+\infty}V(\sqrt{b^2+z''^2})\,dz''}\,db$$

For $\Delta(b) = -\frac{m}{2k\hbar^2}\int_{-\infty}^{+\infty} V(\sqrt{b^2+z^2})dz$

$$f(\vec{k},\vec{k}') = -ik\int_0^\infty b\,db\, J_0(kb\theta)\left[e^{2i\Delta(b)}-1\right]$$

General solution for eikanal approximation

We can regain Born approximation for $kb \ll 1$

Partial waves in eikonal approximation

Quazi-classical treatment $\rightarrow$ many partial waves contribute

$$l_{max} \sim l_{classical} \sim \frac{1}{\hbar} m\vec{v}\cdot\vec{r} = kb \gg 1$$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1) f_l\, P_l(\cos\theta)$$

Useful relationship b/w $P_l(\cos\theta)$ and $J_0(l\theta)$

Legandre eqn for $\theta \ll 1$ and $l \gg 1$

$$\frac{d^2P_l}{d\theta^2} + \frac{1}{\theta}\frac{dP_l}{d\theta} + l^2 P_l = 0 \Rightarrow \text{Bessel equation for } J_0[l\theta]$$

$$P_l(\cos\theta) \approx J_0(l\theta)$$

$$\sum_{l=0}^{l_{max}\to\infty} \xrightarrow[l_{max}]{} k\int db$$

$$f(\theta) \sim k^2\int_0 2b\,db\, f_l\cdot J_0(kb\theta)$$

$$f_l = -\frac{i}{2k}\left(e^{2i\Delta(l/k)}-1\right) = \frac{1}{2ik}\left(e^{2i\delta_l}-1\right)$$

$$\delta_l = \Delta(l/k) = -\frac{m}{2k\hbar^2}\int_{-\infty}^{+\infty} V(\sqrt{l^2/k^2+z'^2})\,dz' \quad \text{for } l < l_{max}$$

$$\delta_l = 0 \quad \text{for } l > l_{max}$$

Low-energy scattering

Low energy $\Rightarrow$ $kR = \sqrt{\frac{2mE}{\hbar^2}}\cdot R \ll 1$
wavelength is $\gg$ potential range

$$e^{ikr} \approx 1 \qquad e^{2i\delta_l} \sim 1$$

Since $\tan\delta_l = \frac{j_l(\rho)}{n_l(\rho)} \cong \frac{\rho^{2l+1}}{(2l+1)!!\,(2l-1)!!} \xrightarrow{\rho \ll 1} 0$

For the slow particle the most important contribution is ~~at~~ $l=0$ $\Rightarrow$ <u>s-wave</u>

This make sense: from the classical point of view, the maximum angular momentum for a particle of given energy $E=\hbar^2k^2/2m$ is

$$L \propto (k\hbar)\cdot R = \hbar\sqrt{l(l+1)}$$

So $l_{max} \sim kR$, for slow waves $kR \sim 0$

For assymptotic solution $r>R$, but $l=0$, k is small ($kr \ll 1$)

$$u_l'' + \Big(\underbrace{\cancel{k^2}}_{=0 \text{ since } k \text{ is small}} - \frac{2m}{\hbar^2}V - \overbrace{\frac{l(l+1)}{r^2}}^{=0}\Big)u_l = 0$$

$$u_0''=0 \quad \Rightarrow \quad u_0 = C_0\,(r-a_s)$$

on the other hand, our "original" asymptotics is still valid

$$u_{\substack{l=0\\k\to0}}(r) \approx C_1\sin\Big(kr - \overbrace{\frac{l\pi}{2}}^{=0} + \delta_{l=0}\Big) = C_1\sin\Big[k\Big(r+\frac{\delta_{l=0}}{k}\Big)\Big] \approx$$

$$\approx C_1 k\Big(r+\frac{\delta_0}{k}\Big) = C_0\,(r-a_s)$$

$$a_s \simeq \lim_{k\to0}\Big(-\frac{\delta_{l=0}}{k}\Big)$$

More formally: for $r \gg R$

$$A_\ell(\rho) = C\,(j_\ell(\rho)\cos\delta_\ell - n_\ell(\rho)\sin\delta_\ell)$$

$\ell=0$ $\quad j_\ell(\rho) = \frac{\sin\rho}{\rho} \qquad n_\ell = -\frac{\cos\rho}{\rho}$

$$u_0(\rho) = rA_0 = \tilde{C}\,[\cos\delta_0 \sin kr + \sin\delta_0 \cos kr] =$$

$$= \left\{\begin{array}{l} kr \ll 1 \\ \sin kr \approx kr \\ \cos kr \approx 1 \end{array}\right\} = \tilde{C}\,[\sin\delta_0 + \cos\delta_0\,(kr)] =$$

$$= \tilde{C}\sin\delta_0\left[1 + \frac{1}{\tan\delta_0}\cdot kr\right] = c_0\,(r - a_s)$$

$$a_s = \lim_{k\to 0}\left(-\frac{\tan\delta_0}{k}\right) \approx \lim_{k\to 0}\left(-\frac{\delta_0}{k}\right) \text{ for small } \delta_0$$

Relate to total scattering cross-section

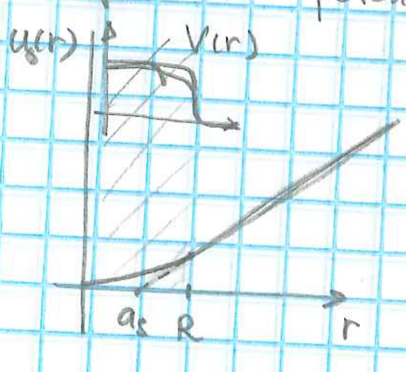
$$\lim_{k\to 0}\sigma_{tot} = \sigma_0 = 4\pi|f_0|^2 \simeq 4\pi\left|\frac{\delta_0}{k}\right|^2 = 4\pi a_s^2$$

$$|f_\ell(\theta)| = \left|\frac{\sin\delta_\ell}{k}\right|$$

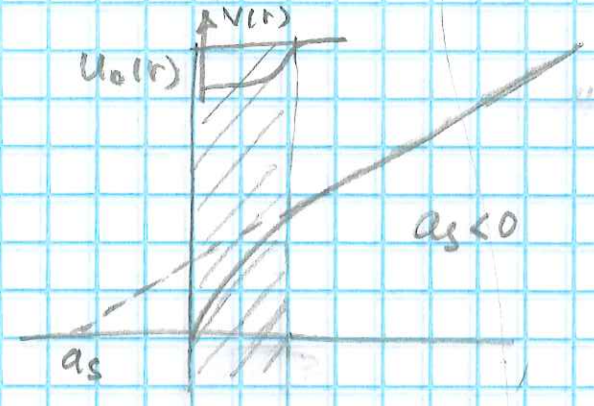
like a hard sphere of radius a_s

a_s called "scattering length", and (almost) fully characterizes low-energy scattering

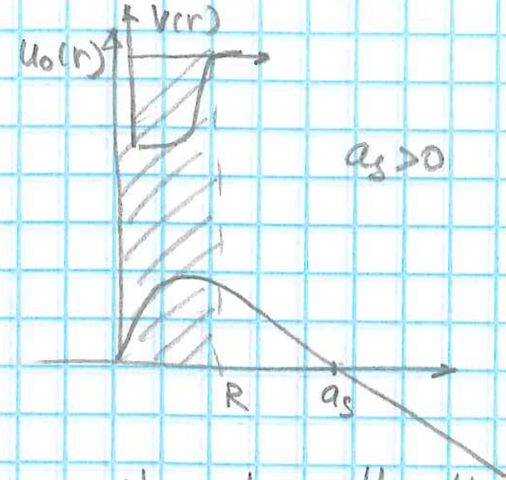
Repulsive potential, $a_s > 0$



Attractive



mildly attractive



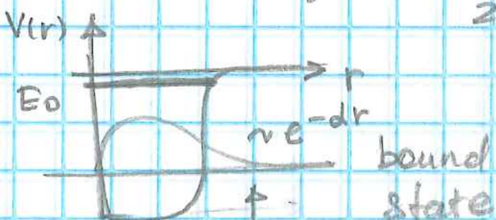
strongly attractive

For attractive potential a_s is very sensitive to the exact details of $V(r)$, and is often too difficult to calculate ab initio

Change of sign of a_s signals bounded state

Scattering length and bound states

Suppose $V(r)$ has a barely bound state with $E_0 = -\frac{\hbar^2\alpha^2}{2m_e} < 0$



exponential decay, but $\alpha \sim 0$

$$\beta = \frac{r}{\psi_0(r)}\frac{d\psi_0}{dr} = \frac{r\frac{d}{dr}e^{-\alpha r}}{e^{-\alpha r}} = -\alpha \cong \lim_{k\to 0}\frac{r u_0'}{u_0} = \lim_{k\to 0} k/\tan\delta_0 = -\frac{1}{a}$$

$\Rightarrow \alpha = -1/a$

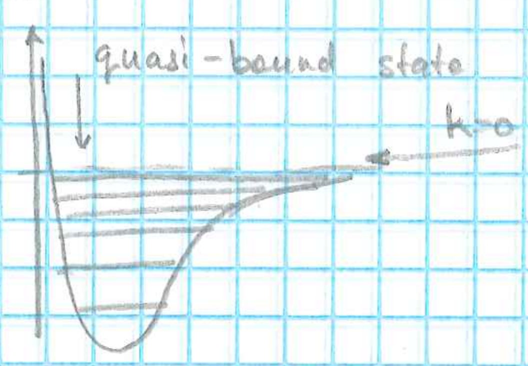
Binding energy $E = \frac{\hbar^2\alpha^2}{2m} = \frac{\hbar^2}{2ma^2}$

Large positive scattering length $a \leftrightarrow$ weakly bound state $E \sim -\frac{\hbar^2}{2ma^2}$

Feshbach Resonanses
scattering from quasi-bound states

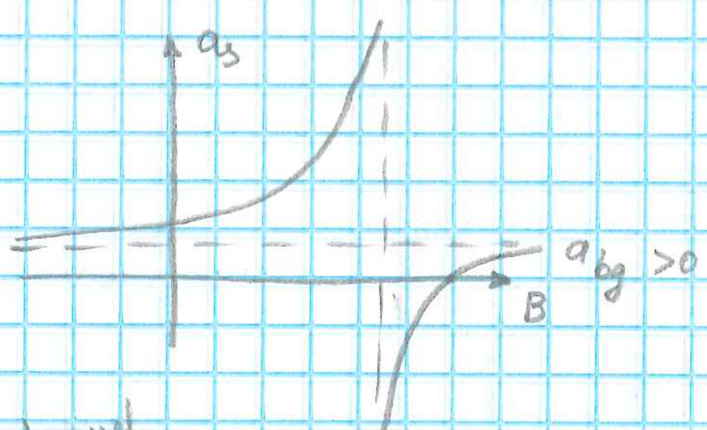
Originally discovered in nuclear scattering

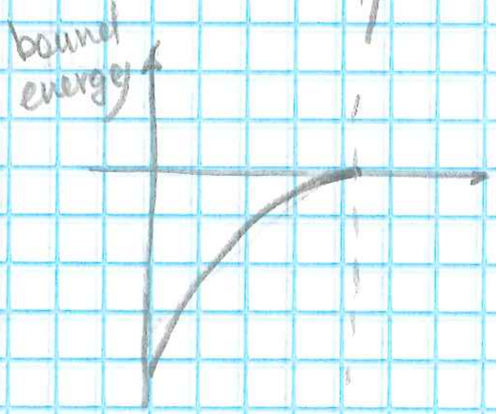
If we can tune the highest bound state from bound into the continuum then we get a resonance in the scattering length



In atoms whe can realize such tunability using Zeeman magnetic field

$$a_s = a_{bg}\left(1 - \frac{\Delta}{B-B_0}\right)$$





$$E_B = -\frac{\hbar^2}{2\mu a^2}$$