# Relativistic quantum mechanics

Natural units $\hbar = c = 1$

$c = 1$ $E, p, m$ — same units. (MeV) ($p$: $\times c$, $m$: $\times c^2$)
time and space have same metric

$\hbar = 1$ $E$ and $\hbar\omega$ — same units

Relativistic energy $E = \sqrt{p^2 + m^2} \xrightarrow{p/m \ll 1} m^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3} + \dots$
non-relativistic energy

Klein - Gordon equation

$$i \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

$$-\frac{\partial^2 \psi}{\partial t^2} = i \frac{\partial}{\partial t} \hat{H} \psi = \hat{H}^2 \psi \qquad \hat{H}^2 = \hat{p}^2 + m^2 = -\nabla^2 + m^2$$

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right] \psi = 0$$ Klein-Gordon eqn for a free particle

Four-vector notation

$x^\mu = (t, \vec{x})$ with metrix $ds^2 = dt^2 - d\vec{x}^2$

any four-vector $a^\mu = (a_0, \vec{a})$
and $a_\mu = (a_0, -\vec{a})$

Inner product $a^\mu a_\mu = a_0^2 - \vec{a} \cdot \vec{a}$

$$\left(\frac{\partial}{\partial t}, \nabla\right) = \frac{\partial}{\partial x^\mu} = \partial_\mu ; \quad \frac{\partial^2}{\partial t^2} - \nabla^2 = \partial_\mu \partial^\mu$$

$$[\partial_\mu \partial^\mu + m^2] \psi = 0$$

To include EM field $\vec{p} \to \vec{p} - e\vec{A}$

$$p^\mu \to p^\mu - eA^\mu$$

$$A^\mu = (\varphi, \vec{A})$$

$$\partial_\mu \to \partial_\mu + ieA_\mu = D_\mu \quad \text{covariant derivative}$$

$$[D_\mu D^\mu + m^2]\,\psi(\vec{r},t) = 0$$

second-order equation: need two initial conditions $(\psi(\vec{r},t)|_{t=0}$ and $\frac{\partial}{\partial t}\psi(\vec{r},t)|_{t=0})$ mixes in the charge of the particle

Break $\Psi(\vec{r},t)$ into two parts

$$\varphi(\vec{r},t) = \frac{1}{2}\left[\psi(\vec{r},t) + \frac{i}{m} D_t \psi(\vec{r},t)\right] \qquad D_t \equiv D_0$$

$$\chi(\vec{r},t) = \frac{1}{2}\left[\psi(\vec{r},t) - \frac{i}{m} D_t \psi(\vec{r},t)\right]$$

two "single" initial conditions $\varphi(t=0)$ & $\chi(t=0)$

KG eqn: $\begin{cases} iD_t\varphi = -\frac{1}{2m} D^2(\varphi+\chi) + m\varphi \\ iD_t\chi = +\frac{1}{2m} D^2(\varphi+\chi) - m\chi \end{cases}$ almost like SE for $\pm m$

Introduce a two-component object

$$\Upsilon(\vec{r},t) = \begin{bmatrix} \varphi(\vec{r},t) \\ \chi(\vec{r},t) \end{bmatrix}$$

$$iD_t \Upsilon = \left[-\frac{1}{2m} D^2(\tau_1 + i\tau_2) + m\tau_3\right]\Upsilon$$

$\tau_{1,2,3}$ – Pauli matices.

We know the form of free particle solutions:

$$\psi(\vec{x},t) \propto e^{i\vec{p}\cdot\vec{x} - iEt} = e^{-ip^\mu x_\mu}$$

$$-p^\mu p_\mu + m^2 = 0 \qquad -E^2 + p^2 + m^2 = 0$$

$$E_p = \pm\sqrt{p^2+m^2}$$

positive and negative solutions!

Problem: four-vector probability current

$$j^\mu = \frac{i}{2m}\left[\psi^* D^\mu \psi - (D^\mu\psi)^*\psi\right]$$

$\partial_\mu j^\mu = 0$ - continuity is ok.

Probability density $\rho(\vec{r},t) = j^0(\vec{r},t) = \frac{i}{2m}\left[\psi^*\frac{\partial\psi}{\partial t} - \left(\frac{\partial\psi}{\partial t}\right)^*\psi\right]$

not necessarily positive.

$\rho(\vec{r},t) = \varphi^*\varphi - \chi^*\chi$ ← charge density

$\varphi(\vec{r},t)$ - wave functions for positive particles
$\chi(\vec{r},t)$ - — " — negative — " —

For the free particle

$$\Upsilon_+(\vec{r},t) = \frac{1}{2(mE_p)^{1/2}}\begin{pmatrix} E_p+m \\ m-E_p\end{pmatrix} e^{-iE_pt + i\vec{p}\vec{r}} \qquad E = E_p$$

$$\Upsilon_-(\vec{r},t) = \frac{1}{2(mE_p)^{1/2}}\begin{pmatrix} m-E_p \\ E_p+m\end{pmatrix} e^{iE_pt + i\vec{p}\vec{r}} \qquad E = -E_p$$

Particle at rest $\quad \chi = 0$ for $E = E_p \quad \rho = 1$
$\qquad\qquad\qquad\quad \varphi = 0$ for $E = -E_p \quad \rho = -1$

Particles with positive charge have positive energy (mass)
Particles with negative charge have negative enery (mass) — anti-particles

# Main postulates of the Dirac theory

1. The theory is formulated in terms of a field, qualitatively represented by an amplitude function $\psi$, in such a way that the statistical interpretation is valid

2. The description of physical phenomena are based on an eqn of motion describing the time evolution of the system or of the field amplitude $\psi$.

3. Superposition principle holds $\rightarrow$ eqns are linear

4. Eqns of motion are consistent with special relativity $\longrightarrow$ covariant form

5. It must be possible to define a probability density that is positive

$$\rho \geq 0$$

and $\int \rho \, d^3\vec{r} = \int \rho' \, d^3\vec{r}'$ Lorentz-invariant

$$\frac{d}{dt} \int \rho \, d^3\vec{r} = 0 \implies \int \rho \, d^3\vec{r} = 1$$

6. It must be consistent with the correspondence principle and in its non-relativistic limit should reduce to the standard form of non-relativistic quantum mechanics.

(Adopted from K. Potamianos "Dirac eqn")

Postulate 2:

Schrodinger eqn - like equation

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi$$ ← first-order in time

To be consistent with SR → coordinate-derivatives also must be the first order

Relativistic energy $E = \sqrt{p^2 + m^2}$

$$\frac{\partial}{\partial t} \to E = p^0$$

$$(p^0 - \sqrt{m^2 + \vec{p}^{\,2}})\psi = 0$$

or $((p^0)^2 - m^2 - \vec{p}^{\,2})\psi = 0$ or $(\partial_\mu \partial^\mu + m^2)\psi = 0$

Klein - Gordon eqn

What if we can define a new operator

$D_\mu = (A\partial_\mu - Bm)$ and

$$D_\mu D^\mu = (\overset{\mu}{A}\partial_\mu - \overset{\mu}{B}m)(A_\mu\partial^\mu - B_\mu m) = \underset{\sim}{A^2}\partial_\mu\partial^\mu + B^2m^2 -$$

$$- (\overset{\mu}{A}B_\mu m\,\partial_\mu + B_\mu\overset{\mu}{A}m\,\partial^\mu) = \partial_\mu\partial^\mu + m^2$$

$$A^2 = 1, \quad B^2 = 1, \quad \overset{\mu}{A}Bm\partial_\mu + BAm\partial^\mu = 0$$

$$D_\mu = i\gamma^\mu\partial_\mu - m$$

Dirac eqn $D_\mu\psi = 0$ $\quad \boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$

or, in a regular form

$$(i\gamma^0\partial_t - i\gamma^i\partial_i - m)\psi = 0 \qquad i = 1,2,3$$

similar to

$$i\partial_t\psi = \hat{H}\psi$$

$(p_0 - \alpha_1 p_1 - \alpha_2 p_2 - \alpha_3 p_3 - \beta m)\,\psi = 0$

and

$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m)\,\psi = 0$

multiply two operators

$$p_0^2 - \left[\alpha_i^2 p_i^2 + (\alpha_i\alpha_j + \alpha_j\alpha_i)\, p_i p_j + (\alpha_i\beta + \beta\alpha_i)\, p_i m + \beta^2 m^2\right] = 0$$

(in summation $i > j$

need to regain $(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2)\,\psi = 0$

$\alpha_i^2 = \beta^2 = 1 \qquad \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \qquad \alpha_i\beta + \beta\alpha_i = 0$

$p_0^2 - \vec{p}^{\,2} - m^2 = 0$

$E^2 - \vec{p}^{\,2} - m^2 = 0$

$\alpha_i\beta = -\beta\alpha_i$

$\alpha_i = -\beta\alpha_i\beta$

$Tr(\alpha_i) = Tr(\alpha_i\beta^2) = -Tr(\beta\alpha_i\beta)$

$= -Tr(\beta^2\alpha_i) = -Tr(\alpha_i) = 0$

$\hat{H}\psi = (\vec{\alpha}\vec{p} + \beta m)\,\psi = E\psi$

$$\boxed{i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\hat{\vec{p}} + \beta m)\psi}$$

we can easily generalize it

$\vec{p} \to \vec{p} - e\vec{A}$

$$(\hat{p}_0 - eA_0) - \vec{\alpha}\,\overbrace{(\vec{p} - e\vec{A})}^{\vec{\pi}} - \beta m = 0$$

$\tilde{\hat{H}}_{EM}$ $\boxed{\hat{H}_{EM} = \vec{\alpha}\vec{\pi} + \beta m + eA_0}$

Because of the requirements $\alpha_i\alpha_j + \alpha_j\alpha_i = \delta_{ij}$
$\alpha_i$ and $\beta$ cannot be just numbers
=> matrices.

since $\psi^\dagger\psi$ = real positive number (density)
matrices must be square

Real eigenvalues of the Hamiltonian =>
matices are hermitian

Minimum possible rank is 4, and (since we need 4 of them)
can be written using Pauli matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Trace must be $=0$

To connect back to covariant notation

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

$$\gamma^\mu = (\beta, \beta\vec{\alpha})$$

Probability density $\rho = \psi^\dagger\psi$
Continuity equation $\partial_\mu j^\mu = 0$
or $\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$

$$\vec{j} = \psi^\dagger\alpha\psi$$

Define $\bar{\psi} = \psi^\dagger\beta = \psi^\dagger\gamma_0$

$$\rho = \psi^\dagger\psi = \psi^\dagger\beta\cdot\beta\psi = \bar{\psi}\beta\psi = \bar{\psi}\gamma_0\psi$$

$$\vec{j} = \psi^\dagger\vec{\alpha}\psi = \psi^\dagger\beta\beta\vec{\alpha}\psi = \bar{\psi}\vec{\gamma}\psi \qquad \underbrace{j^\mu = \bar{\psi}\gamma^\mu\psi}_{\text{four-vector current}}$$

The form of a wave function is a 4-component vector

$$\psi(\vec{r},t) = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_{up} \\ \psi_{down} \end{pmatrix}$$

KG equation - just 2.

spin 1/2 particle is described by a two-component spinor; however, we have twice as many — because we "introduced" negative energy solutions

$$j^\mu = \bar{\psi}\gamma^\mu\psi = \frac{1}{2m}\left(\bar{\psi}\gamma^\mu(m\psi) + (m\bar{\psi})\gamma^\mu\psi\right)$$

$$(\gamma^\mu p_\mu - m)\psi = 0 \quad \text{and} \quad \bar{\psi}(\gamma^\mu p_\mu - m) = 0$$

$$= \frac{1}{2m}\left(\bar{\psi}\gamma^\mu\gamma^\nu p_\nu\psi + \bar{\psi}\gamma^\mu p_\nu\gamma^\mu\psi\right) =$$

$$= \frac{1}{2m}\bar{\psi}\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{\delta_{\mu\nu}} p_\nu\psi = \frac{p_\mu}{2m}\bar{\psi}\psi$$

$$\bar{\psi}\beta\psi = \bar{\psi}\begin{pmatrix} \hat{\mathbb{1}} & 0 \\ 0 & -\hat{\mathbb{1}} \end{pmatrix}\psi = \psi^*_{up}\psi_{up} - \psi^*_{down}\psi_{down}$$

$$j_0 = \frac{E}{m}(\psi^*_\uparrow\psi_\uparrow - \psi^*_\downarrow\psi_\downarrow)$$

$$\vec{j} = \frac{\vec{p}}{m}(\psi^*_\uparrow\psi_\uparrow - \psi^*_\downarrow\psi_\downarrow)$$