Homework 8

7.6

a) $\psi_i = |+\rangle|+\rangle|+\rangle$

$\psi_{ii} = \frac{1}{\sqrt{3}}\left(|+\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|+\rangle + |0\rangle|+\rangle|+\rangle\right)$

$\psi_{iii} = \frac{1}{\sqrt{6}}\left(|+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle + |0\rangle|+\rangle|-\rangle + |0\rangle|-\rangle|+\rangle + \right.$
$\left. + |-\rangle|+\rangle|0\rangle + |-\rangle|0\rangle|+\rangle\right)$

$\hat{S}^2 = (\hat{S}_z^2 + \hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+) = \hat{S}_{z_1}^2 + \hat{S}_{z_2}^2 + \hat{S}_{z_3}^2 + 2\hat{S}_{z_1}\hat{S}_{z_2} + 2\hat{S}_{z_2}\hat{S}_{z_3} +$

$+ 2\hat{S}_{z_1}\hat{S}_{z_3} + \hat{S}_{+1}\hat{S}_{-1} + \hat{S}_{+2}\hat{S}_{-2} + \hat{S}_{+3}\hat{S}_{+3} + \hat{S}_{-1}\hat{S}_{+1} + \hat{S}_{-2}\hat{S}_{+2} + \hat{S}_{-3}\hat{S}_{+3} +$

$+ 2\left(\hat{S}_{+1}\hat{S}_{-2} + \hat{S}_{+1}\hat{S}_{-3} + \hat{S}_{-1}\hat{S}_{+2}\,\hat{S}_{-1}\hat{S}_{+3} + \hat{S}_{+2}\hat{S}_{-3} + \hat{S}_{-2}\hat{S}_{+3}\right) =$

$= \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 + 2(\hat{S}_{z_1}\hat{S}_{z_2} + \hat{S}_{z_2}\hat{S}_{z_3} + \hat{S}_{z_1}\hat{S}_{z_3}) +$

$+ 2(\hat{S}_{+1}\hat{S}_{-2} + \hat{S}_{+1}\hat{S}_{-3} + \hat{S}_{-1}\hat{S}_{+2} + \hat{S}_{-1}\hat{S}_{+3} + \hat{S}_{+2}\hat{S}_{-3} + \hat{S}_{-2}\hat{S}_{+3})$

$\langle\psi_i|\hat{S}^2|\psi_i\rangle = \hbar^2(2+2+2+2\cdot 3) = 12\hbar^2 = \hbar^2 S(S+1) \Rightarrow S=3$
$m_s = 3$

$\langle\psi_{ii}|\hat{S}^2|\psi_{ii}\rangle = \hbar^2\left(2+2+2+2\cdot\frac{3}{3} + 2\cdot\frac{6}{3}\right) = 12\hbar^2 \Rightarrow S=3$
$m_s = 2$

$\hat{S}_+|0\rangle = \sqrt{2}|+\rangle$
$\hat{S}_+|-\rangle = \sqrt{2}|0\rangle$
$\hat{S}_-|0\rangle = \sqrt{2}|-\rangle$
$\hat{S}_-|+\rangle = \sqrt{2}|0\rangle$

$\hat{S}^2|\psi_{iii}\rangle \neq \lambda|\psi_{iii}\rangle$ so $\hat{S}$ is not defined

b) Cases (i) and (2) are impossible in case of anti-symmetric wavefunction

$\psi_{iii}^{(as)} = \frac{1}{\sqrt{6}}\left(|+\rangle|0\rangle|-\rangle - |+\rangle|-\rangle|0\rangle - |0\rangle|+\rangle|-\rangle + |0\rangle|-\rangle|+\rangle - \right.$
$\left. - |-\rangle|0\rangle|+\rangle + |-\rangle|+\rangle|0\rangle\right)$

$\langle\psi_{iii}^{(as)}|\hat{S}^2|\psi_{iii}^{(as)}\rangle = \hbar^2\left(2+2+2 - \frac{1}{6}\cdot 2\cdot 6 - \frac{1}{6}\cdot 2\cdot 6\cdot 2\right) = 0 \quad S=0$
$m_s = 0$

7.8

$\bar{e}$ spin $S = 3/2$ $\Rightarrow$ each electronic state can accomodate $4\ \bar{e}$ $m_s = \pm 1/2, \pm 3/2$

$Z = 10$ 4e for 1s $\boxed{1s^4\, 2s^4\, 2p^2}$

4e for 2s

2e for 2p

Hund's rules $\rightarrow$ * max total spin $S = 3/2 + 3/2 = 3$ (even state)

* and max possible angular momentum $\rightarrow L = 1$

(L=2 is prohibited, since the spatial part must be anti-symmetric.

* and min J ; in this case $J = 3 - 1 = 2$

Final state $^7P_2$

E_0 single-photon electric-field

A1: $\vec{E} = \sqrt{\frac{2\pi\hbar\omega}{V}} \left(\hat{a}\, \vec{e}_\lambda\, e^{i\vec{k}\vec{r} - i\omega t + i\pi/2} + \hat{a}^+ \vec{e}_\lambda^{\,*}\, e^{-i\vec{k}\vec{r} + i\omega t - i\pi/2} \right)$ - single mode

Spatial distribution ~~can be defined~~ differently (i.e $\sin kz$)

~~For a Fock state $|n\rangle$~~

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

a) $\langle n|\vec{E}|n\rangle = \underbrace{E_0 \vec{e}_\lambda e^{i\vec{k}\vec{r} - i\omega t + i\pi/2}}_{\text{not expressly written further}} \langle n|\hat{a}|n\rangle + (E_0 \vec{e}_\lambda e^{i\vec{k}\vec{r} - i\omega t + i\pi/2})^* \langle n|\hat{a}^+|n\rangle$

$= 0$

b) $\vec{E}^2 = \_\_\, \hat{a}^2 + \_\_\, \hat{a}^{+2} + E_0^2 \hat{a}^+\hat{a} + E_0^2 \hat{a}\hat{a}^+ =$

$\langle E^2\rangle = E_0^2 \langle n|(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)|n\rangle = (2n+1) E_0^2 = (2n+1) \cdot \frac{2\pi\hbar\omega}{V}$

c) $\langle n|N|n\rangle = \langle n|\hat{a}^+\hat{a}|n\rangle = n$

$\langle n|N^2|n\rangle = \langle n|\hat{a}^+\hat{a}\,\hat{a}^+\hat{a}|n\rangle = n^2$

$\langle \Delta N^2\rangle = n^2 - n^2 = 0$

Fock state has fixed number of photons (their phase is completely undefined)

A2: Coherent state $|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} |n\rangle \cdot e^{-|\alpha|^2/2}$

$\hat{a}|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\,|n-1\rangle = e^{-|\alpha|^2/2} \cdot \alpha \sum_{n=1} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle$

$n' = n-1$

$= \alpha|\alpha\rangle$

A3:

$$\vec{E} = E_0 \vec{e}_\lambda e^{i\vec{k}\vec{r} - i\omega t + i\pi/2} \hat{a} + h.c.$$

$$\langle\alpha|\vec{E}|\alpha\rangle = \alpha E_0 \vec{e}_\lambda e^{i\vec{k}\vec{r} - i\omega t + i\pi/2} + c.c$$

$$\langle\alpha|E^2|\alpha\rangle = \left(\langle\alpha|\vec{E}|\alpha\rangle\right)^2 \quad \Rightarrow \langle\Delta E^2\rangle = 0$$

$$\langle\alpha|N|\alpha\rangle = \langle\alpha|\hat{a}^+ a|\alpha\rangle = e^{-|\alpha|^2} \sum_{n,n'=0}^{\infty} \frac{\alpha^{*n'}}{\sqrt{n'!}} \frac{\alpha^n}{\sqrt{n!}} \overbrace{\langle n'|\hat{a}^+\hat{a}|n\rangle}^{n\delta_{nn'}} =$$

$$= e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{(|\alpha|^2)^n}{n!} \cdot n = |\alpha|^2 e^{-|\alpha|^2} \sum_{n=\cancel{0}}^{\infty} \frac{(|\alpha|^2)^{n-1}}{(n-1)!} = |\alpha|^2$$

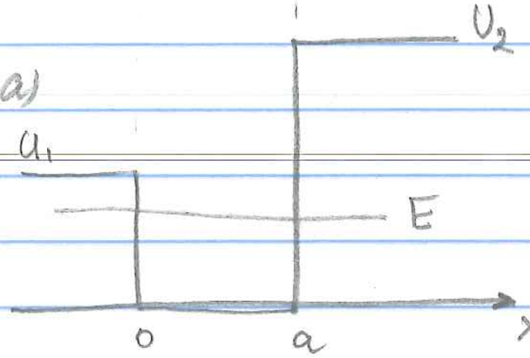
$$\langle\alpha|N^2|\alpha\rangle = e^{-|\alpha|^2} \sum_{n,n'} \frac{(\alpha^*)^{n'}}{\sqrt{n'!}} \frac{(\alpha)^n}{\sqrt{n!}} \overbrace{\langle n'|\hat{a}^+\hat{a}\hat{a}^+\hat{a}|n\rangle}^{n^2\delta_{nn'}} =$$

$$= e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{(|\alpha|^2)^n}{n!} n^2 = |\alpha|^2 e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{(|\alpha|^2)^{n-1}}{(n-1)!} ((n-1)+1)) =$$

$$= |\alpha|^2 e^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{(|\alpha|^2)^{n-1}}{(n-1)!} + |\alpha|^4 e^{-|\alpha|^2} \sum_{n=2}^{\infty} \frac{(|\alpha|^2)^{n-2}}{(n-2)!} = |\alpha|^4 + |\alpha|^2$$

$$\langle\Delta N^2\rangle = \langle N^2\rangle - \langle N\rangle^2 = |\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2$$

Q 1 a)



$$\psi(x) = \begin{cases} A e^{\varkappa_1 x} & x \le 0 \\ B\cos kx + C\sin kx & 0<x<a \\ D e^{-\varkappa_2 (x-a)} & x \ge a \end{cases}$$

$$\varkappa_{1,2} = \sqrt{\frac{2m(U_{1,2}-E)}{\hbar^2}}$$

$$k = \sqrt{2mE/\hbar^2}$$

Boundary conditions

$x=0$ $\begin{cases} A = B & (i) \\ \varkappa_1 A = kC & (ii) \end{cases}$ $x=a$ $\begin{cases} B\cos ka + C\sin ka = D & (iii) \\ -kB\sin ka + Ck\cos ka = -\varkappa_2 D & (iv) \end{cases}$

Substitute (i) and (ii) into:

(iii): $A\cos ka + \frac{\varkappa_1}{k} A \sin ka = A\left(\cos ka + \frac{\varkappa_1}{k}\sin ka\right) = D$

(iv): $kA\left(-\sin ka + \frac{\varkappa_1}{k}\cos ka\right) = -\varkappa_2 D$

divide the two

$$\frac{k(-\sin ka + \varkappa_1/k \cos ka)}{\cos ka + \varkappa_1/k \sin ka} = -\varkappa_2$$

Introducing $\xi_{1,2} = \varkappa_{1,2}/k = \sqrt{\frac{U_{1,2}-E}{E}}$

$$(-\sin ka + \xi_1 \cos ka) = -\xi_2(\cos ka + \xi_1 \sin ka)$$

$$(\xi_1+\xi_2)\cos ka = (1-\xi_1\xi_2)\sin ka$$

$$\tan ka = \frac{\xi_1+\xi_2}{1-\xi_1\xi_2}$$

b) For small $a \to 0$: $\tan(ka) \to 0$
(since $E \leq U_1$ to be a bound state)
~~Thus the equation transforms into~~
$\xi_1 + \xi_2 = 0$, which does not have solutions as ξ_1 and $\xi_2 > 0$

To have at least one bound state at $E \approx U_1$ (i.e $\xi_1 = 0$, $\xi_2 \approx \sqrt{\frac{U_2 - U_1}{U_1}}$)

$$\tan\left[\sqrt{\frac{2mU_1}{\hbar^2}} \cdot a_0\right] = \sqrt{\frac{U_2 - U_1}{U_1}}$$

c) For $U_1 >> U_1$ $\sqrt{\frac{U_2 - U_1}{U_1}} \approx \sqrt{\frac{U_2}{U_1}} >> 1$

$$\sqrt{\frac{2mU_1}{\hbar^2}}\, a_0 \approx \frac{\pi}{2} \quad \text{and} \quad a_0 \approx \frac{\pi}{2}\sqrt{\frac{\hbar^2}{2mU_1}}$$

If this is a qualifier, you may want to double-check your work. So if U_2 is really high, it is like a solid wall

$\Rightarrow$ $B\cos ka + C\sin ka = 0$

$A\cos ka + A\,\varkappa_1/k \sin ka = 0 \Rightarrow \cos ka + \varkappa_1/k \sin ka = 0$

$\therefore$ First bound state : $E \approx U_1$, $\varkappa_1 \approx 0$ $k = \sqrt{\frac{2mU_1}{\hbar^2}}$

Clearly, $\tan ka_0 = \pi/2$ is a solution