5.28

$$\vec{E} = E_0 \vec{e}_z e^{-t/\tau}$$

$$V = -E_0 z\, e^{-t/\tau} \qquad t \geq 0$$

The only non-zero matrix element for $n=2$

$$\langle 100|Ez|210\rangle = \frac{E_0}{4\sqrt{2\pi}\,a_0^3}\, 2\pi \int \frac{r}{a_0}\, e^{-3r/2a_0} \cdot r\cos\theta\; r^2 \sin\theta\, dr\, d\theta =$$

$$= \frac{E_0}{2\sqrt{2}\,a_0^4} \int_0^\infty r^4 e^{-3r/2a_0}\, dr \cdot \int_0^\pi \cos\theta\, \sin\theta\, d\theta =$$

$$= \frac{16\sqrt{2}}{3^5} E_0 a_0 \int_0^\infty x^4 e^{-x} dx = \frac{16\sqrt{2}}{243} E_0 a_0 \cdot 8 = \frac{128\sqrt{2}}{243} E_0 a_0$$

$$c_{210}(t) = \frac{i}{\hbar} \langle 100|Ez|210\rangle \int_0^t e^{-t'/\tau}\, e^{i\omega_{21} t'}\, dt' =$$

$$= -\frac{i}{\hbar} \frac{128\sqrt{2}}{243} E_0 a_0 \; \frac{1 - e^{-t/\tau + i\omega_{21} t}}{i\omega_{21} - 1/\tau}$$

$$P_{210}(t) = |c_{210}(t)|^2 = \frac{E_0^2 a_0^2}{\hbar^2} \cdot \frac{2^{15}}{3^{10}} \; \frac{e^{-2t/\tau} - 2e^{-t/\tau}\cos\omega_{21} t + 1}{\omega_{21}^2 + (1/\tau)^2}$$

for $t \gg \tau$

$$P_{210} \approx \frac{E_0^2 a_0^2}{\hbar^2}\; \frac{2^{15}}{3^{10}}\; \frac{1}{\omega_{21}^2 + (1/\tau)^2}$$

$$\omega_{21} = \frac{E_{210} - E_{100}}{\hbar} = \frac{3e^2}{8a_0\hbar}$$

5.29

$$\hat{H} = \frac{4\Delta}{\hbar^2}\vec{S}_1\vec{S}_2 = \frac{2\Delta}{\hbar^2}\left((\vec{S}_1+\vec{S}_2)^2 - S_1^2 - S_2^2\right)$$

Exact solution : spin addition

$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$

$\hat{S}^2 = \hbar \qquad m_s = \pm\hbar, 0$

States: $|++\rangle, |--\rangle, \boxed{\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)} = \psi_1$

For all three $\Delta E = \Delta$

$S=0 \qquad m_s = 0 \qquad \Delta E = -3\Delta \qquad \psi_0 = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$

Initial state $\psi(t=0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_0)$

$\psi(t) = \frac{1}{\sqrt{2}}(\psi_1 e^{-i\Delta t/\hbar} + \psi_0 e^{3i\Delta t/\hbar}) =$

$= \frac{1}{2}|+-\rangle(e^{-\frac{i\Delta t}{\hbar}} + e^{3\frac{i\Delta t}{\hbar}}) + \frac{1}{2}|-+\rangle(e^{-\frac{i\Delta t}{\hbar}} - e^{\frac{i3\Delta t}{\hbar}}) =$

$= |+-\rangle \cos 2\Delta t\, e^{\frac{i\Delta t}{\hbar}} - |-+\rangle \sin 2\Delta t\, e^{\frac{i\Delta t}{\hbar}}$

$P_{|+-\rangle} = \cos^2 2\Delta t/\hbar$ Clearly, states $|++\rangle$ and $|--\rangle$

$P_{|-+\rangle} = \sin^2 2\Delta t/\hbar$ are not involved $P_{++} = 0$

Perturbation station :

$$C_{|+-\rangle} = 1 - \frac{i}{\hbar}\int_0^t \langle +-|\hat{H}|+-\rangle\, dt = 1 - \frac{i}{2\hbar}\int_0^t \langle \psi_1+\psi_0|\hat{H}|\psi_1+\psi_0\rangle dt$$

$$= 1 - \frac{i}{2\hbar}\int_0^t (\underbrace{\langle\psi_1|\hat{H}|\psi_1\rangle}_{\Delta} + \underbrace{\langle\psi_0|\hat{H}|\psi_0\rangle}_{-3\Delta})\, dt = 1 + \frac{i\Delta}{\hbar}t$$

$$C_{|-+\rangle} = -\frac{i}{\hbar}\int_0^t \langle -+|\hat{H}|+-\rangle\, dt = -\frac{i}{2\hbar}\int_0^t \langle\psi_1-\psi_0|\hat{H}|\psi_1+\psi_0\rangle dt =$$

$$= -\frac{i}{2\hbar}\int_0^t (\underbrace{\langle\psi_1|\hat{H}|\psi_1\rangle}_{\Delta} - \underbrace{\langle\psi_0|\hat{H}|\psi_0\rangle}_{-3\Delta})\, dt = -\frac{2i\Delta t}{\hbar}$$

$P_{-+} = \frac{4\Delta^2 t^2}{\hbar^2} = \sin^2\frac{2\Delta t}{\hbar}\Big|$ for $\Delta t/\hbar \ll 1$, but $P_{+|}(t)$ does not work

5-30

$$\hat{V} = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix}$$

Exact solution

$$i\hbar \dot{c}_1 = V_{12} e^{i\omega_{12} t} c_2 = \gamma e^{i\omega t} e^{-i\omega_{21} t} c_2 = \gamma e^{i(\omega-\omega_{21})t} c_2$$

$$i\hbar \dot{c}_2 = V_{21} e^{i\omega_{21} t} c_1 = \gamma e^{-i\omega t} e^{i\omega_{21} t} c_1 = \gamma e^{-i(\omega-\omega_{21})t} c_1$$

$$\tilde{c}_1 = c_1 e^{-i(\omega-\omega_{21})t} \quad \Rightarrow \quad c_1 = \tilde{c}_1 e^{i(\omega-\omega_{21})t}$$

$$i\hbar \dot{c}_2 = \gamma \tilde{c}_1 \quad \Rightarrow \quad i\hbar \ddot{c}_2 = \gamma \dot{\tilde{c}}_1 = \gamma \left(\dot{c}_1 e^{-i(\omega-\omega_{21})t} - i(\omega-\omega_{21}) c_1 e^{-i(\omega-\omega_{21})t} \right)$$

$$i\hbar \ddot{c}_2 = \gamma \left(\frac{\gamma}{i\hbar} \cancel{e^{i(\omega-\omega_{12})t}} c_2 \cancel{e^{-i(\omega-\omega_{21})t}} - i(\omega-\omega_{21}) \underbrace{c_1 e^{-i(\omega-\omega_{21})t}}_{=\frac{i\hbar}{\gamma}\dot{c}_2} \right)$$

$$i\hbar \ddot{c}_2 = \frac{\gamma^2}{i\hbar} c_2 + \hbar(\omega-\omega_{21}) \dot{c}_2$$

$$\ddot{c}_2 = -\frac{\gamma^2}{\hbar^2} c_2 - i(\omega-\omega_{21})\dot{c}_2 \qquad c_2 = A e^{i\lambda t}$$

$$-\lambda^2 = -\gamma^2/\hbar^2 + \lambda(\omega-\omega_{21})$$

$$\lambda^2 + \lambda(\omega-\omega_{21}) - \gamma^2/\hbar^2 = 0$$

$$\lambda_{1,2} = -\frac{\omega-\omega_{21}}{2} \pm \sqrt{\frac{(\omega-\omega_{21})^2}{4} + \frac{\gamma^2}{\hbar^2}} \equiv \Omega$$

$$c_2 = A_1 e^{i\lambda_1 t} + A_2 e^{i\lambda_2 t}$$

$$c_1 = \frac{i\hbar}{\gamma} \dot{c}_2 e^{i(\omega-\omega_{21})t} = \frac{i\hbar}{\gamma} e^{i(\omega-\omega_{21})t} \left(i\lambda_1 A_1 e^{i\lambda_1 t} + i\lambda_2 A_2 e^{i\lambda_2 t} \right)$$

For $t=0 \Rightarrow A_1 + A_2 = 0$ & $-\frac{\hbar}{\gamma}(\lambda_1 A_1 + \lambda_2 A_2) = 1$

$A_2 = -A_1$

$$\lambda_1 A_1 - \lambda_2 A_1 = (\lambda_1 - \lambda_2) A_1 = -\frac{\gamma}{\hbar} \Rightarrow A_1 = -\frac{\gamma}{\hbar(\lambda_1 - \lambda_2)}$$

$$A_2 = \frac{\gamma}{\hbar(\lambda_1 - \lambda_2)}$$

$$|c_2|^2 = (A_1 e^{i\lambda_1 t} + A_2 e^{+i\lambda_2 t})(A_1 e^{-i\lambda_1 t} + A_2 e^{-i\lambda_2 t}) =$$

$$= A_1^2 + A_2^2 + A_1 A_2\, e^{i(\lambda_1-\lambda_2)t} + A_1 A_2\, e^{-i(\lambda_1-\lambda_2)t} =$$

$$= A_1^2 + A_2^2 + 2A_1A_2 \cos(\lambda_1-\lambda_2)t = \frac{2\gamma^2}{\hbar^2(\lambda_1-\lambda_2)^2}\left(1-\cos(\lambda_1-\lambda_2)t\right)$$

$$= \frac{4\gamma^2}{\hbar^2(\lambda_1-\lambda_2)^2} \sin^2\left[\frac{\lambda_1-\lambda_2}{2} t\right] = \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\Omega t$$

$$|c_1|^2 = 1 - |c_2|^2$$

Perturbation theory

$$c_2 = -\frac{i}{\hbar}\int_0^t V_{12}\, e^{-i\omega_{21}t}\, dt = -\frac{i}{\hbar}\int_0^t \gamma e^{i\omega t - i\omega_{21}t}\, dt =$$

$$= -\frac{i\gamma}{\hbar}\,\frac{e^{i(\omega-\omega_{21})t}-1}{i(\omega-\omega_{21})} = \frac{\gamma}{\hbar(\omega-\omega_{21})}\left(1-e^{i(\omega-\omega_{21})t}\right) =$$

$$= \frac{2\gamma}{\hbar(\omega-\omega_{21})}\, e^{\frac{i(\omega-\omega_{21})t}{2}} \sin\frac{(\omega-\omega_{21})t}{2}$$

$$|c_2|^2 = \frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}\sin^2\frac{(\omega-\omega_{21})t}{2}$$

Comparison: $|\omega-\omega_{21}| \gg \gamma$ case; perturbation is weak, we expect the PT to be valid

$$\Omega = \sqrt{\frac{(\omega-\omega_{21})^2}{4} + \gamma^2/\hbar^2} \approx \frac{\omega-\omega_{21}}{2}$$

so exact solution

$$|c_2|^2 = \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\Omega t \approx \frac{4\gamma^2}{\hbar^2(\omega-\omega_{21})^2}\sin^2\frac{(\omega-\omega_{21})t}{2} \quad \checkmark$$

$|\omega-\omega_{21}| \ll \gamma$

$$\Omega = \sqrt{\frac{(\omega-\omega_{21})^2}{4} + \gamma^2/\hbar^2} \approx \gamma/\hbar$$

$$|c_2|^2 = \frac{\gamma^2}{\hbar^2(\omega-\omega_{21})^2}\sin^2\left(\frac{\gamma}{\hbar}t\right) \longleftarrow \text{not even close!}$$

A1

$$\vec{E} = \mathcal{E} e^{-t^2/\tau^2} \vec{e}_z$$

a) From $n=2$ the only possible transition is to $|210\rangle$ state

$$\hat{V} = -e\mathcal{E} e^{-t^2/\tau^2} \hat{z}$$

$$\langle 100|\hat{V}|210\rangle = \langle 100|-e\mathcal{E}z|210\rangle e^{-t^2/\tau^2} =$$

$$= -e\mathcal{E}a_0 \cdot \frac{128\sqrt{2}}{243} e^{-t^2/\tau^2} \qquad \text{(from 5-28)}$$

$$c_{210} = \frac{i}{\hbar}\left(e\mathcal{E}a_0 \cdot \frac{128\sqrt{2}}{243}\right) \int_{-\infty}^{t} e^{-t'^2/\tau^2} e^{i\omega t'} dt'$$

$$\int_{-\infty}^{t} e^{-t'^2/\tau^2} e^{i\omega t'} dt' = e^{-\frac{\omega^2\tau^2}{4}} \int_{-\infty}^{t} e^{-1/\tau^2 (t' - \frac{\omega\tau^2}{2})^2} dt' \longrightarrow$$

$$\xrightarrow{t\to\infty} \tau\sqrt{\pi}\, e^{-\omega^2\tau^2/4}$$

$$c_{210} = \frac{i}{\hbar} e\mathcal{E}a_0 \frac{128\sqrt{2\pi}}{243} \tau e^{-\omega^2\tau^2/4}$$

$$P_{210} = |c_{210}|^2 = \left(\frac{e\mathcal{E}a_0}{\hbar}\right)^2 \frac{\pi \cdot 2^{15}}{3^{10}} \tau^2 e^{-\omega^2\tau^2/2}$$

b) Yes! time-varying electric field creates magnetic field $[\nabla \times \vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t}]$ which would 'lift the degeneracy for two spin orientations.

a) For a stationary well

$$\psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\frac{\pi n x}{L}\, e^{-iE_n t/\hbar} \qquad E_n = \frac{\pi^2 n^2 \hbar^2}{2mL^2}$$

A naive guess for the wave function in a uniformally moving well may be

$$\tilde{\psi}_n(x,t) = \sqrt{\frac{2}{L}} \sin\frac{\pi n}{L}(x-vt)\, e^{-i\tilde{E}_n t/\hbar} \qquad \tilde{E}_n = E_n \pm \frac{mv^2}{2}$$

since from Galileo relativity principle the probability in moving and stationary cases should be identical with $x \leftrightarrow x-vt$

However, it is easy to check that such naive guess is not correct, and the wave function does not satisfy the Schrödinger equation.

Probably the safest route is to solve the ~~Shro~~ time-dependent Schrödinger equation, having in mind that the space coordinate will enter only in $(x-vt)$ combination

$$\psi(x,t) = f(\overbrace{x-vt}^{z})\, e^{-i\lambda t} \qquad \lambda = E/\hbar$$

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\frac{\partial\psi}{\partial t} = \left[f'\cdot(-v) - i\lambda f\right] e^{-i\lambda t}$$

$$\frac{\partial\psi}{\partial x} = f' e^{-i\lambda t} \qquad \frac{\partial^2\psi}{\partial x^2} = f'' e^{-i\lambda t}$$

$$\cancel{-} i\cancel{\hbar}\,[v f' + i\lambda f] = \cancel{-}\frac{\hbar^{\cancel{2}}}{2m} f''$$

$$f'' - 2\frac{imv}{\hbar} f' + \frac{2m\lambda}{\hbar} f = 0$$

$$f(z) = A e^{\alpha z}$$

$$\alpha^2 - 2\frac{imv\alpha}{\hbar} + \frac{2m\lambda}{\hbar} = 0 \qquad \alpha_{1,2} = \frac{imv}{\cancel{2}\hbar} \pm i\sqrt{\frac{2m\lambda}{\hbar} + \frac{m^2v^2}{\hbar^2}}$$

$$f(z) = A_1 \exp\left\{\left(\frac{imv}{\cancel{2}\hbar} + i\sqrt{\frac{2m\lambda}{\hbar} + \frac{m^2v^2}{\hbar^2}}\right) z\right\} + A_2 \exp\left\{\left(\frac{imv}{\cancel{2}\hbar} - i\sqrt{\frac{2m\lambda}{\hbar} + \frac{m^2v^2}{\hbar^2}}\right) z\right\}$$

Boundary conditions $f(z=0)=0 \qquad f(z=L)=0$

$$f(z) = A e^{imvz/\cancel{2}\hbar} \sin\frac{\pi n z}{L} \quad \text{where} \quad \sqrt{\frac{2m\lambda}{\hbar} + \frac{m^2v^2}{\hbar^2}}\cdot L = \pi n$$

Thus

$$\tilde{\psi}_n(x,t) = \sqrt{\frac{2}{L}}\, e^{\frac{imv(x-vt)}{\hbar}} \sin\frac{\pi n(x-vt)}{L}\, e^{-i\tilde{E}_n t/\hbar}$$

$$\tilde{E}_n = \frac{\pi^2 n^2 \hbar^2}{2mL^2} - \frac{mv^2}{2}$$

or $\tilde{\psi}_n(x,t) = \sqrt{\frac{2}{L}}\, e^{\frac{imvx}{\hbar}} \sin\frac{\pi n(x-vt)}{L}\, e^{-\frac{i\pi^2 n^2 \hbar t}{2mL^2} - \frac{imv^2}{2\hbar}}$

b) Ground state wave function for the moving well

$$\tilde{\varphi}_0(x,t) = \sqrt{\frac{2}{L}}\, e^{i\frac{mv}{\hbar}(x-vt)} \sin\frac{\pi(x-vt)}{L}\, e^{-iE_1 t/\hbar}$$

For the stationary well

$$\varphi_k(x,t) = \sqrt{\frac{2}{L}} \sin\frac{\pi k x}{L}\, e^{-iE_k t/\hbar}$$

The probability to fing the ~~same~~ particle in the k-th state of the stationary well is

$$p_k = \left| \int_0^L \tilde{\varphi}_0^*\Big|_{t=0} \varphi_k\Big|_{t=0} dx \right|^2 = \frac{2}{L}\left| \int_0^L e^{-\frac{imvx}{\hbar}} \sin\frac{\pi x}{L} \sin\frac{\pi k x}{L} dx \right|^2$$

$$= \frac{2}{L}\left[\left(\int_0^L \cos\frac{mvx}{\hbar} \sin\frac{\pi x}{L} \sin\frac{\pi k x}{L} dx \right)^2 + \left(\int_0^L \sin\frac{mvx}{\hbar} \sin\frac{\pi x}{L} \sin\frac{\pi k x}{L} dx \right)^2 \right]$$

c) Clearly that is the x-dependence phase of the wave-function that makes the transition probability non-zero

Thus if $\frac{mvL}{\hbar} \ll 1$ the probability of the transition is small.

That is equivalent to saying that the wavelength of the particle

$\lambda_{part} = \frac{\hbar}{mv} \gg L$, or, alternatively, that

the momentum associated with the well motion is much smaller than the characteristic momentum of each state

$$\frac{mv}{\hbar} \ll \frac{\pi}{L}$$

or $\frac{mv^2}{2} \ll \frac{\pi^2 \hbar^2}{2mL^2}$ same condition