Homework 11

8.1

a) $m_p = 0.94\ GeV$

b) "Natural" length $x \to \frac{x}{c\hbar}$; $1 fm \Leftrightarrow 5.1\cdot 10^{-9} (eV)^{-1}$

$$E \sim p \sim \frac{\hbar}{x} = 1.97\cdot 10^{8}\ eV = 0.2 GeV$$

comparable to the pion mass

c) $[\hbar] = [E]\cdot[T]$

$[c] = [L]/[T]$ $\qquad U = G\frac{mm}{r}$

$[G] = [E]\cdot[L]/[M]^2$ $\qquad E = \frac{mv^2}{2}$

$[E] = [M]\cdot[L]^2/[T]^2 \Rightarrow$

$$[\hbar] = \frac{[M][L]^2}{[T]} \qquad [G] = \frac{[L]^3}{[M][T]^2}$$

$$\frac{[\hbar]}{[G]} = \frac{[M]^2\ [T]}{[L]} = [M]^2/[c] \Rightarrow [M] = \sqrt{\frac{[\hbar][c]}{[G]}}$$

$$M_G c^2 = \sqrt{\frac{\hbar c}{G}}\cdot c^2 = \sqrt{\frac{\hbar c^5}{G}} = 1.2\cdot 10^{19}\ GeV$$

8.10

$$\frac{\partial \rho}{\partial t} = \frac{\partial(\psi^\dagger\psi)}{\partial t} = \frac{\partial \psi^\dagger}{\partial t}\cdot\psi + \psi^\dagger\frac{\partial\psi}{\partial t}$$

$$\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\vec{p}\,\psi - i\beta m\psi = -\vec{\alpha}\,\nabla\psi - i\beta m\psi$$

$$\frac{\partial\psi^\dagger}{\partial t} = -\nabla\psi^\dagger\cdot\vec{\alpha} + im\beta\psi^\dagger$$

$$\frac{\partial\rho}{\partial t} = -\nabla\psi^\dagger\vec{\alpha}\psi + \cancel{im\beta\psi^\dagger\psi} - \psi^\dagger\vec{\alpha}\nabla\psi^\dagger \cancel{- i\beta m\psi^\dagger\psi}$$

$$= -\nabla(\psi^\dagger\vec{\alpha}\psi) = -\nabla\vec{j}$$

$$\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0$$

8.12

$$u_R^{(+)} = \frac{1}{\sqrt{1+\left(\frac{p}{E_p+m}\right)^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{1+\left(\frac{p}{E_p+m}\right)^2}} \begin{pmatrix} u \\ \frac{p}{E_p+m} u \end{pmatrix}$$

$$j^\mu = \bar{\psi} \gamma^\mu \psi = \psi^+ \gamma^0 \gamma^\mu \psi$$

$$j^0 = \psi^+ (\gamma^0)^2 \psi = \psi^+ \psi = \frac{u^+ u + \left(\frac{p}{E_p+m}\right)^2 u^+ u}{1 + \left(p/E_p+m\right)^2} = 1$$

$$\vec{j} = \psi^+ \gamma^0 \vec{\gamma} \psi = \psi^+ \vec{\alpha} \psi = \psi^+ \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \underset{\begin{pmatrix} u \\ \lambda u \end{pmatrix}}{\psi} =$$

$$= \left(u_R^{(+)}\right)^+ \begin{pmatrix} \frac{p}{E_p+m} \vec{\sigma} u \\ \vec{\sigma} u \end{pmatrix} \frac{1}{\sqrt{1+p^2/(E_p+m)^2}} =$$

$$= \frac{1}{(1+p^2/(E_p+m)^2)} \frac{2p}{E_p+m} \left[u^+ \vec{\sigma} u \right]$$

$$(1) \qquad (1\ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$(2) \qquad (1\ 0) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$(3) \qquad (1\ 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$ (or -1 for u_p^- solution)

$\vec{j} = \frac{\vec{p}}{E_p}$ since $\vec{p} = (0, 0, p)$

and $\vec{j} = -\vec{p}/E_p$ for $u_p^{(-)}$ solution

A1

a) $u_{\lambda'}^{+} u_{\lambda} = \varphi_{\lambda'}^{+}\left[1+\frac{(\vec{\sigma}\vec{p})^2}{(E_p+m)^2}\right]\varphi_{\lambda} = \frac{2E_p}{E_p+m}\underbrace{(\varphi_{\lambda'}^{+}\varphi_{\lambda})}_{\delta_{\lambda'\lambda}}$

since $\varphi_{\lambda}, \varphi_{\lambda'}$ are eigenfunctions of $\vec{n}\vec{\sigma}$

b) If $\vec{p}=0$ $\quad u_{\lambda}^{(0)} = \begin{pmatrix}\varphi_{\lambda}\\ 0\end{pmatrix}$

Then $\quad (\vec{\Sigma}\vec{n})\, u_{\lambda}^{(0)} = (\vec{\Sigma}\vec{n})\begin{pmatrix}\varphi_{\lambda}\\ 0\end{pmatrix} = \lambda\,(\vec{\Sigma}\vec{n})\,u_{\lambda}^{(0)}$

$u_{\lambda}^{(0)}$ - eigenfunctions of the spin operator $\vec{\Sigma}\vec{n}$. Thus, if the $\vec{n}\vec{S}$ component of the spin is measured, the eigenvalues are going to be $\pm\frac{1}{2}=\frac{\lambda}{2}$ in the coordinate system where the particle is at rest.

c) The operator $\frac{1}{2}\varphi^{*}\vec{\sigma}\varphi = \varphi^{*}(\frac{1}{2}\vec{\sigma})\varphi$ gives in the rest frame for the particle the average value of its spin

$u^{+}\frac{1}{2}\vec{\Sigma}\,u = \begin{pmatrix}\varphi\\ 0\end{pmatrix}^{+}\left(\frac{1}{2}\vec{\Sigma}\right)\begin{pmatrix}\varphi\\ 0\end{pmatrix} = \varphi^{*}(\frac{1}{2}\vec{\sigma})\varphi$

A1: At $t=0$

$$\psi_R^{(+)}(p)=\frac{1}{\sqrt{V}}\sqrt{\frac{E_p+m}{2E_p}}\begin{bmatrix}1\\0\\P/E_p+m\\0\end{bmatrix}e^{ipz}\qquad \psi_L^{(+)}(p)=\frac{1}{\sqrt{V}}\sqrt{\frac{E_p+m}{2E_p}}\begin{bmatrix}0\\1\\0\\\frac{-p}{E_p+m}\end{bmatrix}e^{ipz}$$

$$\psi_R^{(-)}(p)=\frac{1}{\sqrt{V}}\sqrt{\frac{E_p+m}{2E_p}}\begin{bmatrix}-P/E_p+m\\0\\1\\0\end{bmatrix}e^{ipz}\qquad \psi_L^{(+)}(p)=\frac{1}{\sqrt{V}}\sqrt{\frac{E_p+m}{2E_p}}\begin{bmatrix}0\\P/E_p+m\\0\\1\end{bmatrix}e^{ipz}$$

$$\psi=\frac{1}{\sqrt{V}}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}e^{ipz}$$

$$P_R^{(+)}=\int_V dV\,|\langle\psi^\dagger_{(R)}|\psi\rangle|^2=\int_V dV\cdot\frac{1}{V}\left(\frac{E_p+m}{2E_p}\right)|(a+\frac{P}{E_p+m}c)|^2=$$

$$=\left(\frac{E_p+m}{2E_p}\right)|(a+\frac{P}{E_p+m}c)|^2 \qquad E>0,\ \text{spin up}$$

$$P_R^{(-)}=\left(\frac{E_p+m}{2E_p}\right)|-\frac{P}{E_p+m}a+c|^2 \qquad E<0,\ \text{spin up}$$

$$P_L^{(+)}=\left(\frac{E_p+m}{2E_p}\right)|(b-\frac{P}{E_p+m}d)|^2 \qquad E>0,\ \text{spin down}$$

$$P_L^{(-)}=\left(\frac{E_p+m}{2E_p}\right)|(\frac{P}{E_p+m}b+d)|^2 \qquad E<0,\ \text{spin down}$$

A2 It is possible to simplify one's life by directing z-axis along $\vec{p}$, so that $\vec{p} = (0, 0, p)$

Then it is trivial to show that wavefunctions SN(8.2.22) are the eigenfunctions of the helicity operator $h = \Sigma_z$ with eigenvalues ± 1

The operator $\gamma^0 \Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$

It is easy to check that $\langle \psi^{(+)} | \gamma^0 \Sigma | \psi^{+} \rangle =$

$$= \left[1 - p^2/(E_p+m)^2\right]\left(\frac{E_p+m}{2E_p}\right) = \frac{m}{E_p}$$

and $\langle \psi^{(-)} | \gamma^0 \Sigma | \psi^{-} \rangle = -\left[1 - p^2/(E_p+m)^2\right]\left(\frac{E_p+m}{2E_p}\right) =$

$$= -\frac{m}{E_p}$$

Q1. $V(x) = \frac{\hbar^2 \lambda}{2m} \delta(x)$

a) $E > 0$ $\quad \psi(x) = \frac{1}{\sqrt{2\pi}} e^{\pm ikx} = \begin{cases} \frac{1}{\sqrt{2\pi}} (e^{ikx} + A e^{-ikx}) & x<0 \\ \frac{1}{\sqrt{2\pi}} B e^{ikx} & x>0 \end{cases}$

Boundary conditions

$$1 + A = B$$

$$ik\big(B - (1-A)\big) = \lambda(1+A)$$

$$2ikA = \lambda(1+A) \quad \Rightarrow \quad A = \frac{\lambda}{2ik - \lambda}$$

Thus $\quad \psi(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} (e^{ikx} + A e^{-ikx}) & x<0 \\ \frac{1}{\sqrt{2\pi}} (e^{ikx} + A e^{ikx}) & x>0 \end{cases}$

$$\Rightarrow \psi(x) = \frac{1}{\sqrt{2\pi}} \Big(1 + \underbrace{\frac{\lambda}{2ik-\lambda}}_{f(k)} \cdot e^{ik|x|} \Big)$$

$E < 0$ $\quad \psi(x) = \sqrt{\alpha}\, e^{-\alpha|x|} \qquad \alpha = \sqrt{-\frac{2mE}{\hbar^2}}$

$$-2\alpha = \lambda \qquad \alpha = -\lambda/2$$

bound state for $\lambda < 0$ only

$f(k) = \frac{\lambda}{2ik - \lambda} \quad \Rightarrow$ pole @ $k = -i\lambda/2 = i\alpha$

The pole corresponds to the bound state.

b) 1D Greens function (SN 6.1) $\quad G(x,x') = \frac{e^{ik|x-x'|}}{2ik}$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{2m}{\hbar^2} \int G(x,x') V(x') \psi(x) dx' =$$

$$= \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{\cancel{2m}}{\cancel{\hbar^2}} \frac{1}{2ik} \cdot \frac{\lambda \cancel{\hbar^2}}{\cancel{2m}} \int_{-\infty}^{+\infty} e^{ik|x-x'|} \psi(x') \delta(x') dx' =$$

$$= \frac{1}{\sqrt{2\pi}} e^{ikx} + \frac{\lambda}{2ik} e^{ik|x|} \psi(0) = \frac{1}{\sqrt{2\pi}} \Big(e^{ikx} + \frac{\lambda}{2ik - \lambda} e^{ik|x|} \Big)$$

$$\Uparrow$$

$$\psi(0) = \frac{1}{\sqrt{2\pi}} + \frac{\lambda}{2ik} \psi(0) \Rightarrow \psi(0) = \frac{1/\sqrt{2\pi}}{1 - \lambda/2ik}$$