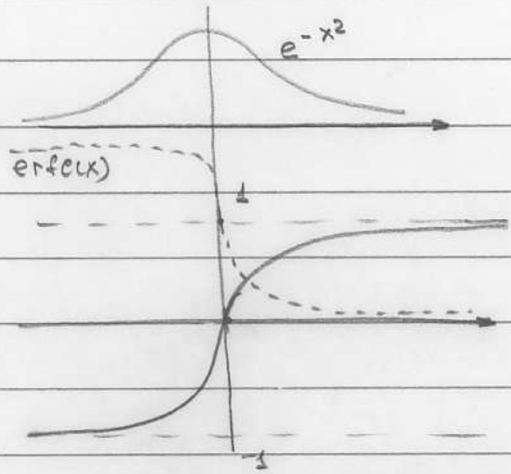
Lecture #9
(2009)

Special function (cont)

3. Error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Sketch



$\mathrm{erf}(0) = 0$; $\mathrm{erf}(+\infty) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-t^2} dt = 1$

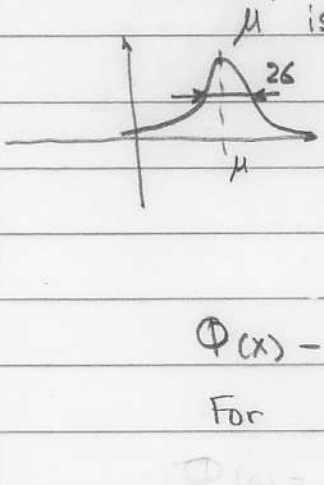
For very small x $\quad e^{-t^2} \approx 1 \quad \mathrm{erf}(x) \approx \frac{2}{\sqrt{\pi}} x$

$$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt = -\,\mathrm{erf}(x)$$

Where is the name "error" come from? From statistics. We will show later that many statistical distributions can be approximated with a normal (Gaussian) distribution for a large sample.

Gaussian probability distribution: $\varphi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

μ is the mean value of x, σ – standard deviation



Then the probability to measure the value of x in the interval $[x_1, x_2]$ is

$$P(x_1, x_2) = \int_{x_1}^{x_2} \varphi(x)\, dx = \Phi(x_2) - \Phi(x_1)$$

$\Phi(x)$ – cumulative distribution function

For the normal distribution $\left[\frac{x-\mu}{\sigma} \to y\right]$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2} dy =$$

$$= \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}(x/\sqrt{2}) \quad ; \quad \text{or} \quad \boxed{\mathrm{erf}(x) = 2\Phi(\sqrt{2}x) - 1}$$

Complimentary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = 1 - \operatorname{erf}(x)$$

Either erf(x) or erfc(x) can be use, depending on convenience.

Power series expansion.

Taylor series (convenient for small x)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{n!} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)n!} =$$

$$= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \dots \right)$$

Asymptotic series (expansion for $x >> 1$)

Convenient to expand erfc(x)

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \longrightarrow$$ want to expand into $\left\{\frac{1}{x}\right\}$ power series.

$$e^{-t^2} = \frac{1}{t} \cdot \left(t e^{-t^2} \right) \quad ; \quad \left(t e^{-t^2} \right) dt = -\frac{1}{2} d\left(e^{-t^2}\right)$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{t} \left(t e^{-t^2} \right) dt = -\frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t} d\left(e^{-t^2}\right) = -\frac{1}{\sqrt{\pi}} \left. \frac{e^{-t^2}}{t} \right|_x^\infty +$$

$$+ \frac{1}{\sqrt{\pi}} \int_x^\infty \left(-\frac{1}{t^2} \right) e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t^2} e^{-t^2} dt =$$

$$= \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} - \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{1}{t^3} \left(t e^{-t^2} dt \right) = \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} - \frac{1}{2\sqrt{\pi}} \frac{e^{-x^2}}{x^3} + \frac{3}{2\sqrt{\pi}} \int_x^\infty \frac{1}{t^4} e^{-t^2} dt.$$

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1.3}{(2x^2)^2} - \frac{1.3.5}{(2x^2)^3} + \dots \right)$$

$$a_n = \frac{(2n-1)!!}{(2x^2)^n} \frac{e^{-x^2}}{x\sqrt{\pi}}$$ series diverges.

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2x^2} \xrightarrow{n \to \infty} \infty$$

Let's estimate the error of our expansion

After the two steps of the integration by parts

$$erfc(x) = \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} - \frac{1}{2\sqrt{\pi}} \frac{e^{-x^2}}{x^3} + \frac{3}{2\sqrt{\pi}} \int_x^\infty \frac{1}{t^4} e^{-t^2} dt$$

Remainder $\left| \int_x^\infty \frac{1}{t^4} e^{-t^2} dt \right| \leqslant \left| \frac{1}{x^5} \int_x^\infty t e^{-t^2} dt \right| = \frac{1}{2x^5} e^{-x^2}$

Thus: $\left| erfc(x) - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} - \frac{1}{2\sqrt{\pi}} \frac{e^{-x^2}}{x^3} \right| < \frac{3}{4\sqrt{\pi}} \frac{e^{-x^2}}{x^5}$

So infinite sum diverges, but a finite sum approximates the function very well for large x.

For N number of terms in the asymptotic series

$$\left| erfc(x) - \sum_{n=0}^{N} \frac{e^{-x^2}}{\sqrt{\pi}\, x} \frac{(2n-1)!!}{(2x^2)^n} \right| \leq \frac{1}{\sqrt{\pi}} \frac{(2N+1)!!}{2^{N+1}} \frac{e^{-x^2}}{x^{2N+3}}$$

In regular power series to increase the accuracy for each x we should let the number of terms to go to infinity $\lim_{N\to\infty} \left(f(x) - \sum_{n=0}^{N} a_n x^n \right) = 0$

In asymtotic series the number of terms is fixed, and the approximation becomes more accurate if $x \to \infty$ (or $x \to 0$)

$$\lim_{x\to\infty} \left(f(x) - \sum_{n=0}^{N} \varphi_n(x) \right) \to 0$$

Often $\lim_{x\to\infty} \frac{\left| f(x) - \sum_{n=0}^{\infty} \varphi_n(x) \right|}{\varphi_N(x)} \to 0$

Another example of the asymtrotic series:
incomplete Gamma function

$$\Gamma(p,x) = \int_x^\infty t^{p-1} e^{-t}\, dt$$

$$\int_x^\infty t^{p-1} e^{-t}\, dt = -t^{p-1}e^{-t}\Big|_x^\infty + (p-1)\int_x^\infty t^{p-2}e^{-t}dt = x^{p-1}e^{-x} + (p-1)x^{p-2}e^{-x} +$$

$$+ (p-1)(p-2)\int_x^\infty t^{p-3}e^{-t}dt = \sum_{n=0}^{N}(p-1)(p-2)\dots(p-n-1)\,x^{p-n-1}e^{-x} + R_N(x)$$

$$|R_N(x)| < (p-1)\dots(p-n-2)\frac{e^{-x}}{x^{n+2-p}}$$

It is easy to see that this series diverges for any x,
but $\left|\Gamma(p,x) - \sum_{n=0}^{N}(p-1)(p-2)\dots(p-n-1)\,x^{p-n-1}e^{-x}\right| \Big/ x^{p-n-1}e^{-x} \sim \frac{1}{x} \xrightarrow{x\to\infty} 0$

Assymptotic expansion of the Gamma function
(Stirling's formula)

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}dx = \int_0^\infty e^{-x+p\ln x}\,dx = \left\{\begin{array}{l} x = p + y\sqrt{p} \\ dx = \sqrt{p}\,dy \\ x=0 \Rightarrow y = -\sqrt{p} \end{array}\right\} =$$

$$= \int_{-\sqrt{p}}^\infty \exp\left[-p - y\sqrt{p} + p\ln(p+y\sqrt{p})\right]\cdot\sqrt{p}\,dy$$

For $p \gg 1$ $\quad \ln(p+y\sqrt{p}) = \ln p + \ln\left(1+\frac{y}{\sqrt{p}}\right) \approx \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p}$

$$\Gamma(p+1) = \int_{-\sqrt{p}}^\infty \exp\left[p\ln p + \cancel{\sqrt{p}\,y} - \frac{y^2}{2} - p - \cancel{y\sqrt{p}}\right]\sqrt{p}\,dy =$$

$$= \sqrt{p}\cdot e^{p\ln p - p}\int_{-\sqrt{p}}^\infty e^{-y^2/2}dy = \sqrt{p}\,e^{p\ln p - p}\left[\int_{-\infty}^{+\infty} e^{-y^2/2}dy - \right.$$

$$\left. - \cancel{\int_{-\infty}^{-\sqrt{p}} e^{-y^2/2}dy}\right] \approx \sqrt{2\pi p}\; p^p e^{-p} \qquad \text{Stirling's formula}$$

neglect for large p

$$\boxed{\Gamma(p+1) = p^p e^{-p}\sqrt{2\pi p}\left(1 + \frac{1}{12p} + \frac{1}{288p^2} + \dots\right)}$$