Lecture # 8
2009

Special functions.

1. Gamma function

Factorial is determined as a product:

$n! = n(n-1)(n-2) \ldots 2\cdot 1$ ($0! = 1$ by definition)

Let's find the integral representation for $n!$

$$\int_0^\infty e^{-\alpha x}\,dx = -\frac{1}{\alpha} e^{-\alpha x}\Big|_0^\infty = \frac{1}{\alpha}$$

If we differenciate both sides with respect to α $\left(\frac{\partial}{\partial\alpha}\right)$

$$\frac{\partial}{\partial\alpha}\left[\int_0^\infty e^{-\alpha x}dx\right] = \int_0^\infty (-x)\,e^{-\alpha x}dx = -\frac{1}{\alpha^2};\quad \boxed{\int_0^\infty x e^{-\alpha x}dx = \frac{1}{\alpha^2}}$$

Differenciate one more time

$$\frac{\partial}{\partial\alpha}\left[\int_0^\infty x e^{-\alpha x}dx\right] = \int_0^\infty (-x^2)\,e^{-\alpha x}dx = -\frac{2}{\alpha^3};\quad \boxed{\int_0^\infty x^2 e^{-\alpha x}dx = \frac{2}{\alpha^3}}$$

If we repeat the procedure n times:

$$\int_0^\infty x^n e^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}} \overset{\alpha=1}{\Longrightarrow} \boxed{\int_0^\infty x^n e^{-x}dx = n!}$$

Notice that now $0! = 1$ is obvious $\quad 0! = \int_0^\infty e^{-x}dx = 1.$

The integral representation allows to generalize the factorial $\Rightarrow$ Gamma function $\Gamma(p)$

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}dx \quad \text{for any } p$$

If p is integer $\Gamma(p) = (p-1)!$

Recurrence relationship

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}dx = \underbrace{\cancel{-x^p e^{-x}\Big|_0^\infty}}_{=0} + p\int_0^\infty x^{p-1}e^{-x}dx = p\cdot\Gamma(p)$$

$$\boxed{\Gamma(p+1) = p\,\Gamma(p)}$$

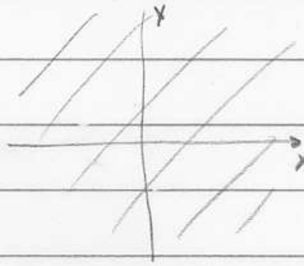
By using $\Gamma(p+1) = p\Gamma(p)$ one needs to know $\Gamma(p)$ for $0<p<1$ to find the values of the Gamma function anywhere else.

Alternative representation

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx = \left\{ \begin{array}{l} x = t^2 \\ dx = 2t\,dt \end{array} \right\} = 2\int_0^\infty t^{2p-1} e^{-t^2} dt$$

Special case $\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-t^2} dt = \int_{-\infty}^{+\infty} e^{-t^2} dt$ Poisson integral

How to calculate? $\Gamma(\tfrac{1}{2}) = \int_{-\infty}^{+\infty} e^{-x^2} dx$; $\Gamma(\tfrac{1}{2}) = \int_{-\infty}^{+\infty} e^{-y^2} dy$



$$\left[\Gamma(\tfrac{1}{2})\right]^2 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\; e^{-x^2+y^2} = \left\{ \begin{array}{l} \text{switch to} \\ \text{polar coordinates} \end{array} \right\} =$$

$$= \int_0^{2\pi} d\varphi \int_0^\infty e^{-r^2} r\,dr = 2\pi \cdot \frac{1}{2}\int_0^\infty e^{-r^2} dr^2 = \pi$$

$$\boxed{\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}}$$

Then $\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \frac{3\sqrt{\pi}}{4}$

In general $\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!}{2^n} \cdot \sqrt{\pi}$

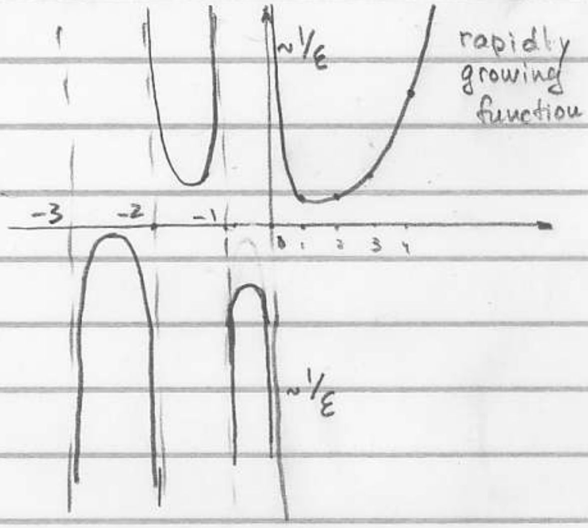
Another important relation

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

What if $p<0$? We will again use $\Gamma(p+1) = p\Gamma(p)$ to see what happens at negative p.

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1) = \frac{1}{p(p+1)}\Gamma(p+2) = \dots$$

Sketch of the Gamma function



1. For positive integer p: $\Gamma(p) = (p-1)!$
2. Gamma function has singularities at $p = 0, -1, -2 \dots$ all negative integer p

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$

Let's find the asymtotic behavior near the singularities.

a) Singularity at $p=0$: $p = \varepsilon$ $|\varepsilon| < 1$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon) \approx \frac{1}{\varepsilon}\Gamma(1) = \frac{1}{\varepsilon}$$

b) Singularity at $p=-1$; $p = -1+\varepsilon$ $|\varepsilon| < 1$

$$\Gamma(-1+\varepsilon) = \frac{1}{(-1+\varepsilon)}\Gamma(\varepsilon) \approx -\frac{1}{\varepsilon}$$

c) Singulatity at $p=-2$ $p = -2+\varepsilon$ $|\varepsilon| < 1$

$$\Gamma(-2+\varepsilon) = \frac{1}{(-2+\varepsilon)}\Gamma(-1+\varepsilon) \approx \frac{1}{2\varepsilon}$$

$$\Gamma\left(-\tfrac{1}{2}\right) = \frac{1}{-\frac{1}{2}}\Gamma\left(\tfrac{1}{2}\right) = -2\sqrt{\pi}$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{1}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}\Gamma\left(\tfrac{1}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\tfrac{5}{2}\right) = \frac{1}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}\Gamma\left(\tfrac{1}{2}\right) = -\frac{8}{15}\sqrt{\pi}$$

Alternative representation of the Gamma function

1. Infinite limit

$$\Gamma(p) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \dots n}{p(p+1)(p+2)\dots(p+n)}\, n^p \qquad p \neq 0, -1, -2 \dots$$

2. Infinite product

$$\frac{1}{\Gamma(p)} = p\, e^{\gamma p} \prod_{n=1}^{\infty}\left(1+\frac{p}{n}\right)e^{-p/n}$$

$\gamma = 0{,}577216$ Euler - Mascheroni constant

2. Beta function

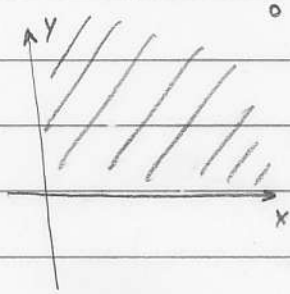
$$B(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1}\, dx \qquad p>0,\ q>0$$

Beta function is symmetric $\quad B(p,q) = B(q,p) \quad [x \to 1-x]$

Beta function can be expressed using Gamma functions:

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$\Gamma(p) = 2\int_0^\infty x^{2p-1} e^{-x^2}\, dx \quad ; \quad \Gamma(q) = 2\int_0^\infty y^{2q-1} e^{-y^2}\, dy$$



$$\Gamma(p)\,\Gamma(q) = 4\int_0^\infty dx \int_0^\infty dy \; x^{2p-1} y^{2q-1} e^{-x^2-y^2} =$$

$$= 4\int_0^{\pi/2} d\varphi \int_0^\infty r\,dr \; [r\cos\varphi]^{2p-1} [r\sin\varphi]^{2q-1} e^{-r^2} =$$

$$= \underbrace{2\int_0^\infty r^{2p+2q-1} e^{-r^2}\, dr}_{\Gamma(p+q)} \cdot \underbrace{2\int_0^{\pi/2} (\cos\varphi)^{2p-1} (\sin\varphi)^{2q-1}\, d\varphi}_{B(p,q)}$$

$$B(p,q) = 2\int_0^{\pi/2} (\cos\varphi)^{2p-1} (\sin\varphi)^{2q-1}\, d\varphi = \left\{\begin{array}{l} t = \sin^2\varphi \\ dt = 2\sin\varphi\cos\varphi\, d\varphi \end{array}\right\} =$$

$$= \int_0^{\pi/2} [\cos^2\varphi]^{p-1} [\sin^2\varphi]^{q-1} [2\sin\varphi\cos\varphi\, d\varphi] = \int_0^1 (1-t)^{p-1} t^{q-1}\, dt = B(q,p) = B(p,q)$$

Thus: $\quad \Gamma(p)\,\Gamma(q) = \Gamma(p+q)\, B(p,q) \qquad$ Q.E.D.

Alternative representation of $B(p,q)$

$$B(p,q) = 2\int_0^{\pi/2} (\sin\varphi)^{2p-1} (\cos\varphi)^{2q-1}\, d\varphi$$

Another useful representation

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \left\{ \begin{array}{l} 1-x = \frac{1}{1+y} \\ x = \frac{y}{1+y}, \quad dx = \frac{dy}{(1+y)^2} \end{array} \right\} =$$

$$= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p-1}} \cdot \frac{1}{(1+y)^{q-1}} \cdot \frac{dy}{(1+y)^2} = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy$$

So little more on $\Gamma(p)\Gamma(1-p) = \overset{=1}{\cancel{\Gamma(1)}} \cdot B(p,1-p) = \int_0^\infty \frac{y^{p-1}}{1+y}\,dy = \frac{\pi}{\sin \pi p}$

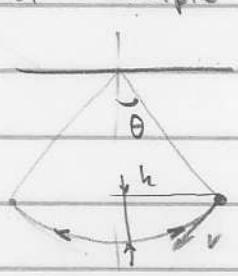
we will be able to do this integral when we talk about contour integrals later.

Examples:

a) $$\int_0^1 \frac{x^4\,dx}{\sqrt{1-x^2}} = \int_0^1 x^4 (1-x^2)^{-1/2}\,dx = \left\{ \begin{array}{l} x^2 = t \\ 2x\,dx = dt \end{array} \right\} = \frac{1}{2}\int_0^1 t^{3/2}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{2}B\left(\frac{5}{2},\frac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{1}{2}\,\frac{3/4\,\Gamma^2(1/2)}{2!} = \frac{3\pi}{16}$$

b. Simple pendulum.



Kinetic energy: $K = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\omega^2 = \frac{1}{2}m\ell^2(\dot\theta)^2$

Potential energy: $U = mgh = mg\ell(1-\cos\theta)$

Energy conservation:

$$\frac{1}{2}m\ell^2(\dot\theta)^2 + mg\ell(1-\cos\theta) = \text{const} = U_0$$

Small oscillations $\theta \ll 1$, $(1-\cos\theta) \approx \theta^2$

Large oscillations – special case: pendulum starts from $\theta = 90°$

In this case $U_0 = mg\ell$

$$\frac{1}{2}\cancel{m}\ell^{\cancel{2}}(\dot\theta)^2 = \cancel{m}g\cancel{\ell}\cos\theta \qquad \Rightarrow \dot\theta = \sqrt{\frac{2g}{\ell}}\sqrt{\cos\theta}$$

$$dt = \sqrt{\frac{\ell}{2g}}\,\frac{d\theta}{\sqrt{\cos\theta}} \qquad T = 4\int_0^{\pi/2}\sqrt{\frac{\ell}{2g}}\,\frac{d\theta}{\sqrt{\cos\theta}} = 4\sqrt{\frac{\ell}{2g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{2\ell}{g}}\,B\left(\frac{1}{2},\frac{1}{4}\right)$$

$$= 7.4163\sqrt{\frac{\ell}{g}}$$