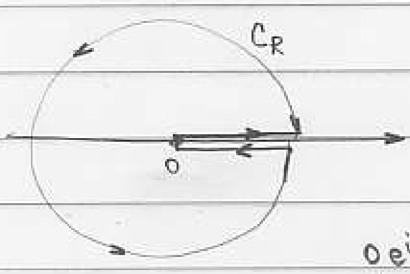
5) $\displaystyle\int_0^\infty \frac{\sqrt{x}}{(x+1)(x+2)}\,dx$

Sometimes, it is important to keep in mind the branches of the complex numbers, i.e. that $z, ze^{i2\pi}, ze^{i4\pi}$ correspond to the same point of the complex plane. However $\sqrt{z}$ and $\sqrt{ze^{i2\pi}} = \sqrt{z}\,e^{i\pi}$ are different points!



$$\oint_C \frac{\sqrt{z}}{(z+1)(z+2)}\,dz = \int_0^R \ldots + \int_{C_R} \ldots + \int_{Re^{i2\pi}}^{0e^{i2\pi}} \ldots$$

$$\int_{C_R} \frac{\sqrt{z}}{(z+1)(z+2)}\,dz \sim \frac{1}{\sqrt{R}} \xrightarrow{R\to\infty} 0$$

$$\int_{Re^{i2\pi}}^{0e^{i2\pi}} \frac{\sqrt{z}}{(z+1)(z+2)}\,dz = \int_R^0 \frac{\sqrt{xe^{i2\pi}}}{(x+1)(x+2)}\,dx = -e^{i\pi}\int_0^R \frac{\sqrt{x}}{(x+1)(x+2)}\,dx$$

$$\oint_C \frac{\sqrt{z}}{(z+1)(z+2)}\,dz = \int_0^\infty \frac{\sqrt{x}}{(x+1)(x+2)}\,dx \cdot \left(1-e^{i\pi}\right) = 2\int_0^\infty \frac{\sqrt{x}}{(x+1)(x+2)}\,dx = 2\pi i \sum_{\substack{z=-1\\ z=-2}} \operatorname{Res} f(z)$$

$$\operatorname*{Res}_{z=-1} \frac{\sqrt{z}}{(z+1)(z+2)} = i \qquad \operatorname*{Res}_{z=-2} \frac{\sqrt{z}}{(z+1)(z+2)} = -\sqrt{2}\cdot i \qquad = 2\pi(\sqrt{2}-1)$$

$$\int_0^\infty \frac{\sqrt{x}}{(x+1)(x+2)}\,dx = \pi(\sqrt{2}-1)$$

6) Let's finally finish the proof that

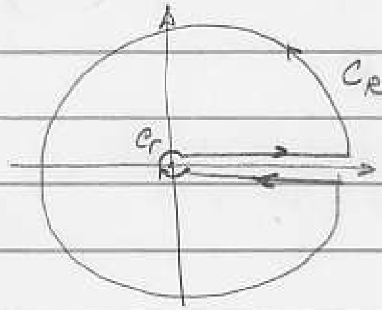
$$\Gamma(p)\Gamma(1-p) = B(p,1-p) = \pi/\sin p\pi$$

Let's use the following representation for B-function

$$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx \Rightarrow B(p,1-p) = \int_0^\infty \frac{x^{p-1}}{1+x}\,dx$$

Let's again take advantage of branching of the complex numbers. $z \to ze^{i2\pi}$ $\quad z^{p-1} \to z\,e^{i2\pi p - i2\pi} = z\,e^{i2\pi p}$

Therefore, we can again use the "full circle" contour as before. Also, if $0<p<1$, the function has a pole at $x=0$, so we may want to "walk around" the origin as well.



$$\oint_C \frac{z^{p-1}}{(1+z)}dz = \int_r^R \ldots + \int_{C_R} \ldots + \int_{Re^{i2\pi}}^{re^{i2\pi}} \ldots + \int_{C_r} \ldots$$

$$\int_{C_R} \frac{z^{p-1}}{1+z}dz \propto \frac{R^p}{R} = \frac{1}{R^{1-p}} \xrightarrow[0<p<1]{R\to\infty} 0$$

$$\int_{C_r} \frac{z^{p-1}}{1+z}dz = \left\{ \begin{array}{l} z = re^{i\varphi} \\ dz = ire^{i\varphi}d\varphi \end{array} \right\} = \int_{2\pi}^{0} \frac{r^{p-1}e^{i(p-1)2\pi} \cdot ire^{i\varphi}d\varphi}{1+\cancel{re^{i\varphi}}}$$

$$= r^p \int_{2\pi}^{0} ie^{ip2\pi}d\varphi \xrightarrow{r\to 0} 0$$

$$\int_{Re^{i2\pi}}^{re^{i2\pi}} \frac{z^{p-1}}{(1+z)}dz = \int_{+\infty}^{0} \frac{x^{p-1}e^{2ip\pi}}{1+x}dx = -e^{2pi\pi}\int_0^{+\infty} \frac{x^{p-1}}{1+x}dx$$

$$\oint_C \frac{z^{p-1}}{1+z}dz = \left(1-e^{2ip\pi}\right)\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = 2\pi i \operatorname*{Res}_{z=-1=e^{i\pi}} \frac{z^{p-1}}{1+z} = -2\pi i e^{ip\pi}$$

Thus $$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = -2\pi i \frac{e^{ip\pi}}{1-e^{2ip\pi}} = \pi \frac{2i}{e^{ip\pi}-e^{-ip\pi}} = \frac{\pi}{\sin p\pi}$$

7. Depending on the integral, the integration contour may be very different from a circle or semicircle we used before. Let's consider, for example

$$\int_{-\infty}^{+\infty} \frac{e^{px}}{1+e^x}dx \qquad 0<p<1$$

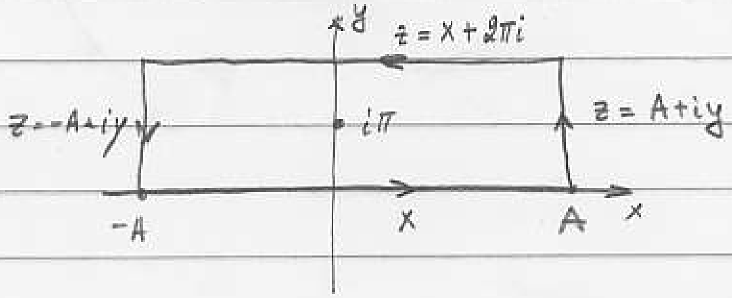
Poles: $1+e^z=0$ $\quad z = i\pi \pm 2\pi i n \quad n=0,1,2\ldots$ – Infinite # of poles.

Moreover, if we choose to integrate over a large arc

$z = Re^{i\varphi} = R\cos\varphi + iR\sin\varphi$

$e^z = \underline{\underline{e^{R\cos\varphi}}} \cdot e^{iR\sin\varphi}$ diverges at large R when $\cos\varphi > 0$

Thus for large $|z|$ e^z is finite only if z is imaginary. That is why we will use a very different contour



$$\oint_C \frac{e^{pz}}{1+e^z}\,dz = \int_{-A}^{A}\ldots + \int_{A}^{A+2\pi i}\ldots + \int_{A+2\pi i}^{-A+2\pi i}\ldots + \int_{-A+2\pi i}^{-A}\ldots$$

$$\int_{A}^{A+2\pi i} \frac{e^{pz}}{1+e^z}\,dz = \int_0^{2\pi} \frac{e^{pA}\,e^{-\pi p y}}{1+e^{A}e^{-\pi y}}\,i\,dy \propto$$

$$\propto \frac{1}{e^{(1-p)A}} \xrightarrow{A\to\infty} 0$$

$$\int_{-A+2\pi i}^{-A} \frac{e^{pz}}{1+e^z}\,dz = \int_{2\pi}^{0} \frac{e^{-pA}\,e^{-\pi p y}}{1+\cancel{e^{-A}e^{-\pi y}}}\,i\,dy \propto e^{-pA} \xrightarrow{A\to\infty} 0$$

$$\int_{A+2\pi i}^{-A+2\pi i} \frac{e^{pz}}{1+e^z}\,dz = \int_{A}^{-A} \frac{e^{px}\cdot e^{2\pi i p}}{1+e^{x}\underbrace{e^{2\pi i}}_{=1}}\,dx = -e^{2\pi i p}\int_{-A}^{A}\frac{e^{px}}{1+e^x}\,dx$$

$$\oint_C \frac{e^{pz}}{1+e^z}\,dz = \left(1-e^{2\pi i p}\right)\int_{-\infty}^{+\infty}\frac{e^{px}}{1+e^x}\,dx = 2\pi i \operatorname*{Res}_{z=i\pi}\left(\frac{e^{pz}}{1+e^z}\right)$$

$$\operatorname*{Res}_{z=i\pi}\frac{e^{pz}}{1+e^z} = \lim_{z=i\pi}\frac{e^{pz}}{1+e^z}(z-i\pi) = \lim_{\substack{\varepsilon\to 0\\ z=i\pi+\varepsilon}} \frac{e^{ip\pi}\cdot e^{p\varepsilon}}{\underbrace{1+e^{i\pi}e^{\varepsilon}}_{1-e^{\varepsilon}}}\cdot\varepsilon = \lim_{\varepsilon\to 0}\frac{\varepsilon\, e^{i\pi p}\cdot e^{p\varepsilon}}{\cancel{1}-\cancel{1}-\cancel{\varepsilon}} =$$

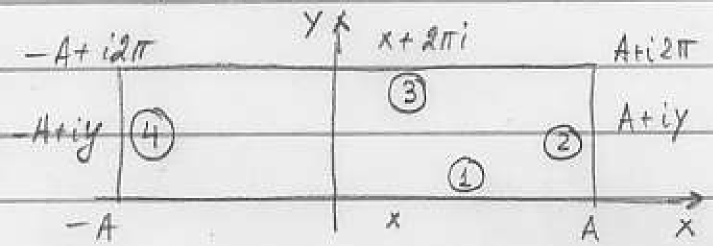
$$= -\,e^{i\pi p}$$

$$\int_{-\infty}^{+\infty}\frac{e^{px}}{1+e^x}\,dx = -2\pi i\,\frac{e^{ip\pi}}{1-e^{2\pi i p}} = \pi\,\frac{2i}{e^{\pi i p}-e^{-\pi i p}} = \frac{\pi}{\sin p\pi}.$$

If you compare two last integrals

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}\,dx \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{e^{py}}{1+e^{y}}\,dy$$

it is easy to notice that they are the same if $x = e^{y}$. We can also check that two contours we use transform one into another under this transformation



Transformation $w = e^{z}$

$z = x+iy$

1. $z = x \qquad [-\infty, +\infty]$ or $[-A, A]$

$w = u+iv = e^{x} \qquad [e^{-A}, e^{A}] \Rightarrow [0, \infty]$

2. $z = A+iy \qquad y = [0, 2\pi]$

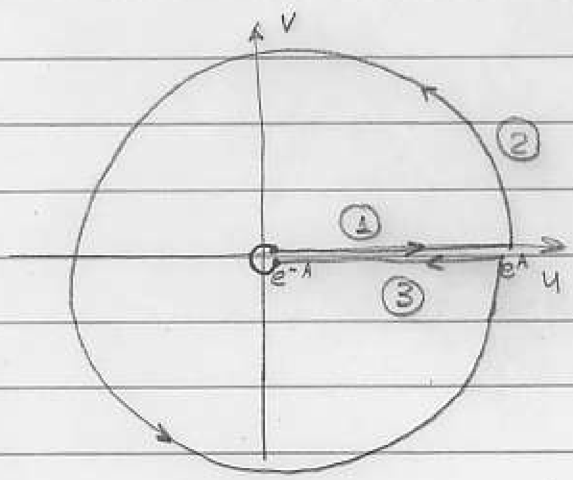
$w = e^{A+iy} = e^{A} \cdot e^{iy}$

3. $z = x + i2\pi \qquad [A, -A]$

$w = e^{x} \underbrace{e^{i2\pi}}_{=1} \qquad [e^{A}, e^{-A}]$

4. $z = -A+iy \qquad y = [0, 2\pi]$

$w = e^{-A} e^{iy}$



This is an example of mapping one contour into another.

So we can use complex functions to map one part of the complex plane into another