Lecture #19
2009

Complex functions.

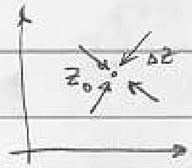
We will use the following notation

$z = x + iy$ — a complex number $\quad z = re^{i\varphi} \quad r = \sqrt{x^2+y^2} \quad \tan\varphi = \frac{y}{x}$

A function of a complex argument

$f(z) = u(x,y) + iv(x,y)$ where u, v are real functions

$f(z)$ is <u>analytic</u>, if it has a unique derivative (at certain point or inside the region)

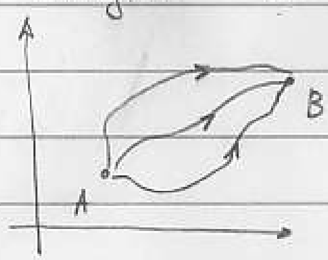


$f'(z) = \lim\limits_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$ is independent on the choice of Δz

if $f(z)$ is analytic.

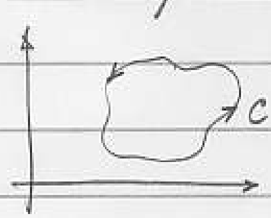
Previously we have shown that if $f(z) = u(x,y) + iv(x,y)$ is analytic, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Integrals of the complex functions



In the complex plane all integrals are <u>contour</u> integrals, since there are many "trajectories" between two points of the complex plane.

Cauchy theorem



If $f(z)$ is analytic inside the closed contour C, then $\oint\limits_C f(z)\,dz = 0$

To prove that we will have to recall Green's theorem:

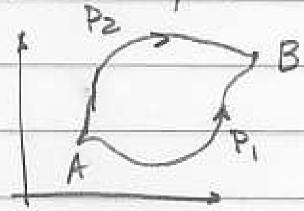
$$\oint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Then:

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy) = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx) =$$

$$= \iint_S \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{=0 \text{ for analytic func}} dx\,dy + i \iint_S \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0 \text{ for analytic func.}} dx\,dy = 0 \qquad \text{Q.E.D.}$$

Cauchy theorem means that all path integrals of an analytic function are equal.

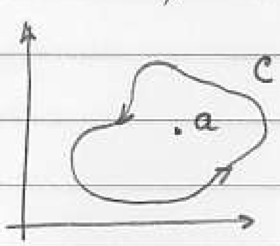


$$\int_{P_1} f(z)\,dz - \int_{P_2} f(z)\,dz = \oint_{P_1,-P_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{P_1} f(z)\,dz = \int_{P_2} f(z)\,dz.$$

Cauchy theorem also gives us a new calculational tool for evaluating contour integrals.

Cauchy integral formula

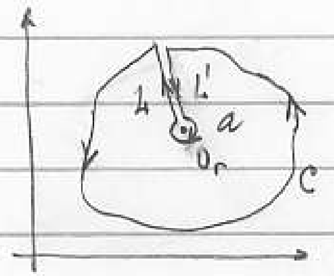


If $f(z)$ is analytic within the contour C, and a is inside C, then

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\,dz$$

Proof : $\frac{f(z)}{z-a}$ is analytic everywhere inside C , except for $z=a$

Let's modify our contour to exclude $z=a$



New contour $C' = C + L + O_r + L'$

O_r is a small circle around $z=a$ of radius r

Straight lines L and L' overlap (and give equal and opposite contributions)

$C' \xrightarrow{r\to 0} C$

$$\oint_{C'} \frac{f(z)}{z-a}\,dz = 0 = \oint_{C} \frac{f(z)}{z-a}\,dz + \cancel{\int_{L} \frac{f(z)}{z-a}\,dz} + \oint_{O_r} \frac{f(z)}{z-a}\,dz + \cancel{\int_{L'} \frac{f(z)}{z-a}\,dz}$$

Let's look carefully at O_r

It is described by $z = a + re^{i\varphi}$ $\quad \varphi = [0, 2\pi]$

$f(z)$ is analytic at a, so $\lim_{r\to 0} f(z) = f(a)$

and $dz = d(a + re^{i\varphi}) = ire^{i\varphi}\,d\varphi$ (since r is constant)

Thus

$$\lim_{r\to 0} \oint_{O_r} \frac{f(z)}{z-a}\,dz = \lim_{r\to 0} \int_{2\pi}^{0} \frac{f(z)}{\cancel{re^{i\varphi}}}\, i\cancel{re^{i\varphi}}\,d\varphi = \lim_{r\to 0} i \int_{2\pi}^{0} f(z)\,d\varphi =$$

$$= i\int_{2\pi}^{0} f(a)\,d\varphi = -2\pi i\, f(a)$$

Thus $\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$

or $\quad f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz$

We will use this result to calculate the values of contour integrals for different complex functions.

Reminder 1: singularity is a point where $f(z)$ is not analytic.

Reminder 2: $f(z)$ can be decomposed into a Taylor series only inside the region where it is analytic. Outside of this region it has to be decomposed into a Laurent series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots$$

the coefficients of the Lauren series can be expressed in a form of a contour integral

$$\underbrace{a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz}_{} \; ; \quad b_n = \frac{1}{2\pi i} \oint f(z)(z-z_0)^{n-1} dz$$

notice: this definition is equivalent to standard $a_n = \frac{1}{n!} \frac{d^n f}{dz^n}\Big|_{z=z_0}$

Classification of the singularities.

a) Suppose that z_0 is a singularity, then if the Laurent series expanded around $z=z_0$ has only $b_1 \neq 0$ (and all other $b_i = 0$) then z_0 is a simple pole

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$$

In practice it often means that $f(z)$ diverges at z_0, but $\lim_{z \to z_0} f(z) \cdot (z-z_0) = b_1$ is finite

b) If the Laurent series contain $b_n \neq 0$, but b_i for $i > n$ are all zeros, then $z = z_0$ is the pole of an order n

$$f(z) = \frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n+1}} + \ldots + a_0 + a_1(z-z_0) + \ldots$$

In practice it often means that $\lim_{z \to z_0} f(z)(z-z_0)^n < \infty$, but all $\lim_{z \to z_0} f(z)(z-z_0)^m \to \infty$ for $m < n$

c) If infinite number of b terms is present, $z = z_0$ is called essential singularity

Examples:

1) $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$ two simple poles at $z = \pm i$

2) $f(z) = \frac{1}{\sin\left(z - \frac{\pi}{2}\right)}$ diverges at $z = \pi/2$

but $\lim \frac{z - \pi/2}{\sin(z - \pi/2)} = 1 \implies z = \pi/2$ is a simple pole

3) $f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$ $z = \pm 2i$ are the poles of the second order

4) $f(z) = \ln z \rightarrow z = 0$ is essential singularity

Residue of a complex function

For a simple pole $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} f(z)(z - z_0)$

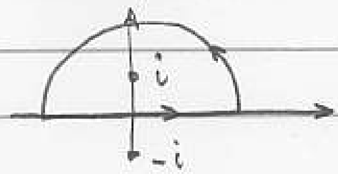
The value of the residue gives the value of a contour integral around the singularity.

Indeed $\oint_C f(z)\,dz = \oint_C \frac{f(z)(z-z_0)}{(z-z_0)}$

since $f(z)(z-z_0)$ is analytic anywhere inside C, then according to the Cauchy integral formula

$$\oint_C f(z)\,dz = 2\pi i \lim_{z \to z_0} f(z)(z-z_0) = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

Example: $f(z) = \frac{1}{z^2+1}$ $\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} \frac{(z-i)}{(z^2+1)} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$



$$\oint \frac{dz}{z^2+1} = 2\pi i \cdot \operatorname*{Res}_{z=i} f(z) = \pi$$

For the higher-order poles

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{(n-1)}}{dz^{n-1}} \left[f(z)(z-z_0)^n \right]$$

Residue theorem

If $f(z)$ is analitic inside C except for a number of singularities $\{z_m\}$, then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_m \operatorname*{Res}_{z=z_m} f(z)}$$

Example: $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ we can turn this integral to a contour integral

$\theta = [0, 2\pi] \longrightarrow z = e^{i\theta}$ integrated over a circle of radius 1

$$\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$dz = i\,e^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = -i\,\frac{dz}{z}$$

$$a+b\cos\theta = a+\frac{b}{2}\left(z+\frac{1}{z}\right) = \frac{1}{z}\left(\frac{b}{2}z^2+az+\frac{b}{2}\right)$$

$$I = -i \oint_{|z|=1} \frac{dz}{\frac{b}{2}z^2+az+\frac{b}{2}} = -\frac{2i}{b}\oint \frac{dz}{z^2+\frac{2a}{b}z+1}$$

$$z_{1,2} = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2}-1} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$$

$$I = -\frac{2i}{b}\oint_{|z|=1} \frac{dz}{\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)\left(z+\frac{a-\sqrt{a^2-b^2}}{b}\right)}$$

the value of the integral depends on a, b

Let's take $a=5$, $b=3$ $\quad z_{1,2} = -3, -1/3$

$$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = -\frac{2i}{3}\oint_{|z|=1} \frac{dz}{(z+3)(z+1/3)} = -\frac{2i}{3}\cdot 2\pi i \operatorname*{Res}_{z=-1/3} f(z) =$$

$$= \frac{4\pi}{3}\,\frac{1}{8/3} = \frac{\pi}{2}$$