Lecture # 17
2009

Time-dependent PDE - wave equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \quad ; \quad \text{in 1D} \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

This equation describes oscillatory and wave motion in many different systems.

For example → running waves

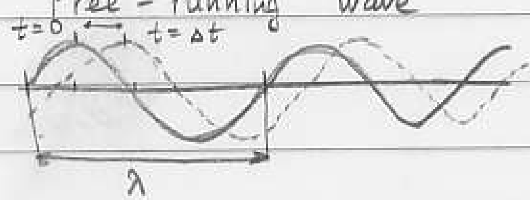
$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\right) = 0 \quad \Rightarrow \quad \left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right) u = 0$$

so any function $F(x \pm vt)$ is a solution of the wave equation

Plane waves: $u(x,t) = A\cos(kz - \omega t)$ - propagates at $+x$ direction
and $u(x,t) = A\cos(kz + \omega t)$ - propagates at $-x$ direction

here ω is a frequency (angular frequency) of the wave
$k = \frac{\omega}{v}$ is a wave-vector; $k = \frac{2\pi}{\lambda}$ where λ is the wavelength

Free-running wave



$$\Delta x = \frac{\omega \Delta t}{k} = v\Delta t$$

v is a phase velocity of the wave

We will focus our attention on the waves propagating inside a string or a surface with given boundaries, and defined boundary conditions.

In this case one can solve the wave equation using the separation of variables

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad u = X(x)\cdot T(t)$$

$$\frac{X''}{X} = \frac{1}{v^2}\frac{\ddot{T}}{T} = -k^2 \qquad \text{to obtain oscillatory solutions}$$

$$X'' = -k^2 X \qquad \ddot{T} = -k^2 v^2 T$$

possible solutions — possible solution

$X(x) = \sin kx$, $\cos kx$ — $T(t) = \sin kvt$, $\cos kvt$

The general solution consist of all possible combinations

For each problem we will have to define <u>boundary conditions</u> and <u>initial conditions</u>

Example 1: The ends of a strings are fixed, so that $u(x=0,t) = u(x=\ell,t) = 0$

Thus $X(0) = X(\ell) = 0 \Rightarrow \sin\frac{\pi n x}{\ell}$

$$k_n = \frac{\pi n}{\ell}, \quad \underbrace{\omega_n = \frac{\pi v n}{\ell}}_{\text{possible oscillation frequencies.}}$$

Thus $u(x,t) = \sum_n \sin\frac{\pi n x}{\ell}\left(A_n \cos\omega_n t + B_n \sin\omega_n t\right)$

The coefficients A_n and B_n are defined from the initial conditions (notice, that you'll need two since the wave equation is the second order PDE in time)

Initial conditions define for a string define <u>initial displacement</u> and <u>initial velocity</u> of the string

a) If we pluck the string: $u(x,t=0) = f(x)$ (displacement)

and the string is not moving: $\frac{\partial u}{\partial t}(x,t=0) = 0$ (velocity)

$$\frac{\partial u}{\partial t}(x,t) = \sum_n \sin\frac{\pi n x}{\ell}\left(-\omega_n A_n \sin\omega_n t + \overbrace{\cancel{\omega_n B_n \cos\omega_n t}}^{=0}\right) \quad \text{if } \frac{\partial u}{\partial t}(t=0) = 0$$

Thus

$$\sum_n A_n \sin\frac{\pi n x}{\ell} = f(x) \quad - \text{ Fourier series}$$

For example

Initial condition: $f(x) = \begin{cases} \frac{2h}{\ell}x & 0 < x < \ell/2 \\ \frac{2h}{\ell}(\ell - x) & \ell/2 < x < \ell \end{cases}$

in one of the homeworks

$$f(x) = \frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{\ell}$$

Thus

$$u(x,t) = \frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{\ell}\cos\frac{(2n+1)\pi v t}{\ell}$$

Oscillations are only on odd harmonics: $\omega_n = \frac{(2n+1)\pi v}{\ell}$

That follows from the symmetry of the initial cond-s.

(You never expect to find a node at $x = \ell/2$)

b) If the string is struck:

zero initial displacement $\quad u(x, t=0) = 0$

$$\frac{\partial u}{\partial t}(x, t=0) = g(x)$$

In this case the solution is $u(x,t) = \sum_n B_n \sin\frac{\pi n x}{\ell} \sin\omega_n t$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \omega_n \sin\frac{\pi n x}{\ell} = g(x) = \sum_{n=1}^{\infty} C_n \sin\frac{\pi n x}{\ell}$$

Example: very sharp hit in the middle: $v(x) = v_0 \cdot \ell\, \delta(x - \ell/2)$

$$C_n = \frac{2}{\ell}\int_0^{\ell} g(x) \sin\frac{\pi n x}{\ell}\,dx = \frac{2}{\ell}\int_0^{\ell} v_0 \ell\, \delta(x-\ell/2) \sin\frac{\pi n x}{\ell}\,dx =$$

$$= 2v_0 \sin\frac{\pi n}{2} = 2v_0 (-1)^k \qquad n = (2k+1) = 1, 3, 5 \ldots$$

$$B_n = C_n/\omega_n = 2v_0(-1)^n / (\pi n v/\ell) = \frac{2v_0 \ell}{\pi v(2k+1)}(-1)^k \quad \text{for } n = 2k+1$$

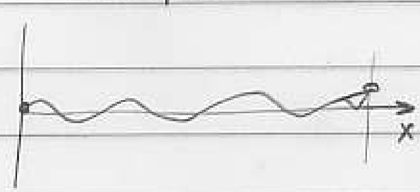
$$u(x,t) = \sum_{k=0}^{\infty} \frac{2v_0 \ell}{\pi v(2k+1)}(-1)^k \sin\frac{\pi(2k+1)x}{\ell} \sin\frac{\pi v(2k+1)t}{\ell}$$

Notice the difference in the contributions of boundary and initial conditions: the boundary conditions define the spectrum of possible harmonics ω_n

(they define the instrument)

Initial conditions define which harmonics are excited, and what are their amplitudes → but they cannot change the frequencies.

Example: one end of the string is unbound



$$u(x=0, t) = 0 \qquad \frac{\partial u}{\partial x}(x=\ell, t) = 0$$

Then $X(x) = \sin kx$, but $X'(x=\ell) = k\cos k\ell = 0$

$$k\ell = \frac{\pi}{2} + \pi n$$

$$X_n(x) = \sin\frac{\pi x}{\ell}\left(n+\tfrac{1}{2}\right) \qquad k_n = \frac{\pi}{\ell}\left(n+\tfrac{1}{2}\right) \quad \omega_n = \frac{\pi v}{\ell}\left(n+\tfrac{1}{2}\right)$$

two unbound ends: $\frac{\partial u}{\partial x}(x=0,t) = \frac{\partial u}{\partial x}(x=\ell,t) = 0$

$$X(x) = \cos kx \Rightarrow X'(x=\ell) = -k\sin k\ell = 0 \qquad k\ell = \pi n$$

$$X_n(x) = \cos\frac{\pi n x}{\ell} \qquad k_n = \frac{\pi n}{\ell}, \quad \omega_n = \frac{\pi n v}{\ell}$$

Vibration of a membrane (2D, Cartesian geometry)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = -k^2$$

$$u = X(x)\, Y(y)\, T(t)$$

$$\underbrace{\frac{X''}{X}}_{=-k_1^2} + \underbrace{\frac{Y''}{Y}}_{=-k_2^2} = \frac{1}{v^2}\frac{\ddot{T}}{T} \qquad \ddot{T} = -(k_1^2+k_2^2)v^2 T$$

$$\omega = \sqrt{k_1^2+k_2^2}\, v$$

Solving for $X(x)$ and $Y(y)$ boundary conditions, we find $\{k_x\}_n$ and $\{k_y\}_m$; thus the oscillation frequencies are

$$\omega_{nm} = \sqrt{k_{xn}^2 + k_{ym}^2}\cdot v$$

Cylindrical geometry

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = -k^2$$

$$u(\vec{r}, t) = F(\vec{r})\cdot T(t)$$

$$\underline{\nabla^2 F = -k^2 F} \quad ; \quad \ddot{T} = -k^2 v^2 T$$

Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} + k^2 F = 0$$

Solution : $F(\vec{r}) = J_n(kr)\,(A_n \cos n\varphi + B_n \sin n\varphi)$

For a fixed boundary of the membrane $(u(r=a, \varphi, t)=0)$

$$u(r,\varphi,t) = \sum_{n,i} J_n(\alpha_i^{(n)} r/a)\,(A_{ni}\cos n\varphi + B_n \sin n\varphi)\,(C_n \cos\omega_{ni}t + D_n \sin\omega_{ni}t)$$

where $\omega_{ni} = k_{ni}v = \underline{\alpha_i^{(n)}\, v/a}$

Example : the drum is struck in the middle

$$u(r,\varphi,t=0) = 0$$

$$\frac{\partial u}{\partial t}(r,\varphi,t=0) = v_0 a^2 \delta(r)$$

Then : $n=0$ (no φ dependence), $C_n = 0$ $\qquad \int \delta(\vec{r})\, r\,dr\,d\varphi = 1$

$$u(r,\varphi,t) = \sum_i J_0\left(\alpha_i^{(0)}\frac{r}{a}\right) D_i \sin\omega_i t \qquad \Rightarrow \int \delta(\vec{r})\, r\,dr = \frac{1}{2\pi}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_i \omega_i J_0\left(\alpha_i^{(0)}\frac{r}{a}\right) D_i = v_0 a^2 \delta(\vec{r})$$

$$D_i\omega_i \int_0^a J_0\left(\alpha_i^{(0)}\frac{r}{a}\right) J_0\left(\alpha_j^{(0)}\frac{r}{a}\right) r\,dr = D_j\omega_j \frac{1}{2} J_1^2(\alpha_j^{(0)})a^2 = v_0 a^2 \int J_0\left(\alpha_j^{(0)}\frac{r}{a}\right)\delta(\vec{r})\,r\,dr$$

$$D_j \omega_j \cdot \frac{1}{2} J_1^2(\alpha_j^{(0)}) \cancel{a^2} = v_0 a^2 \frac{1}{2\pi} \; J_0\left(\alpha_j^{(0)} \frac{r}{a}\right)\Big|_{r=0} = v_0 \cancel{a^2} \frac{1}{2\pi}$$

$$D_j = \frac{v_0}{\pi \omega_j} \; \frac{1}{J_1^2(\alpha_j^{(0)})}$$

$$u(r, \varphi, t) = \frac{v_0 a}{\pi v} \sum_i \frac{1}{\alpha_i^{(0)} J_1^2(\alpha_i^{(0)})} \; J_0\left(\alpha_i^{(0)} \frac{r}{a}\right) \sin\left(\alpha_i^{(0)} \frac{vt}{a}\right)$$