# Orthogonality and normalization of the Bessel functions

Orthogonality conditions we know:

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0 \quad \text{for} \quad n \neq m$$

$$\int_{-1}^{1} \sin \pi n x \, \sin \pi m x \, dx = 0 \quad \text{for} \quad n \neq m$$

notice that $\pi n$ & $\pi m$ are the zeros of $\sin x$

Bessel function has similar orthogonality properties:
If $\alpha_i^{(p)}$ are the zeros of $J_p(x)$, then

$$\int_0^1 J_p(\alpha_i^{(p)} r)\, J_p(\alpha_j^{(p)} r)\, r\,dr = 0 \quad \text{for} \quad i \neq j$$

To prove that one will use a "standard" method of writing the Bessel equation for two different $J_p(\alpha r)$

$$x^2 y'' + x y' + (\alpha_i^2 x^2 - p^2) y = 0 \;\Rightarrow\; x \frac{d}{dx}(x y') + (\alpha_i^2 x^2 - p^2) y = 0,$$

multiplying both equations by $J_p(\alpha_j r)$ and $J_p(\alpha_i r)$ correspondingly, and subtracting them — see Boas Ch 12 - 19 for details.

Normalization

$$\int_0^1 r J_p(\alpha_i r)\, J_p(\alpha_j r)\, dr = \delta_{ij} \cdot \frac{1}{2} J_{p+1}^2(\alpha_i) = \delta_{ij} \cdot \frac{1}{2} J_{p-1}^2(\alpha_i) = \delta_{ij} \cdot \frac{1}{2} J_p'^2(\alpha_i)$$

If the integration limit is different $r \to r/a$

$$\int_0^a r J_p\left(\alpha_i \frac{r}{a}\right) J_p\left(\alpha_j \frac{r}{a}\right) dr = \frac{1}{2} a^2 J_{p+1}^2(\alpha_i) = \frac{1}{2} J_{p-1}^2(\alpha_i) = \frac{1}{2} a J_p'(\alpha_i)$$

# Partial Differential equations

1. Laplace equation: $\nabla^2 u = 0$
2. Helmholtz equation: $\nabla^2 u + ku = 0$

   ] linear steady-state PDE

3. Poisson's equation: $\nabla^2 u = f(\vec{r})$ non-linear steady-state PDE
4. Diffusion equation: $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$
5. Wave equation: $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$
6. $-\frac{\hbar^2}{2m}\nabla^2 \Psi + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}$

   ] time-dependent linear PDE

We will start with the Laplace equation (Helmholtz equation gets very similar treatment)

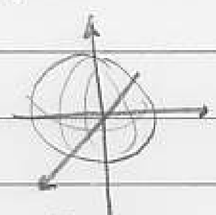
$\nabla^2 u = 0$ ⟶ describes electric potential with no charges

— " — gravitational — " — — " — masses

— " — temperature distribution with no heat sources

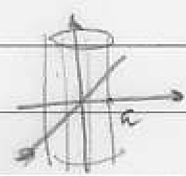
— " — velocity distribution of fluids with no sources or sinks

For each particular situation the solution of the equation will depend on <u>boundary</u> <u>conditions</u>



$\nabla^2 \varphi = 0$

$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$

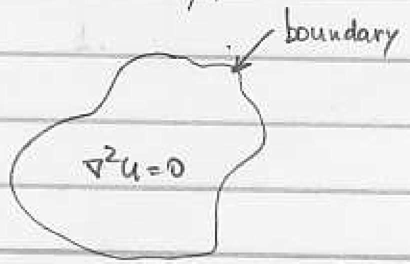


$\nabla^2 \varphi = 0$

$\varphi(r) = \frac{\lambda}{2\pi\varepsilon_0}\ln r/a$

For example, the symmetries of the boundary conditions will define the preferred geometry of the problem.

Main types of the boundary conditions




(1) Dirichlet BC: the value of the solution on the boundary is specified

$$u(\vec{r})\Big|_{\text{on the boundary}} = f(\vec{r})$$

Example: grounded metal surface: $\varphi(\vec{r})\Big|_{\text{surface}} = 0$
or given temperature distribution on the surface

$$u(\vec{r})\Big|_{\text{surface}} = T_0(\vec{r})$$

(2) Neumann BC: the value of the normal derivative of the solution is specified

$$\frac{\partial}{\partial \vec{n}} u\Big|_{\text{boundary}} = (\nabla u \cdot \vec{n})\Big|_{\text{boundary}} = f(\vec{r})$$

Example: thermoisolated boundary (no heat flow) $(\nabla u \cdot \vec{n})\Big|_{\text{surf.}} = 0$

(3) Cauchy BC: combination of (1) and (2)

Laplace eqn in <u>Cartezian coordinates</u> (2D case)

$\nabla^2 u = 0 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Separating the variables

$$u(x,y) = X(x)\,Y(y)$$

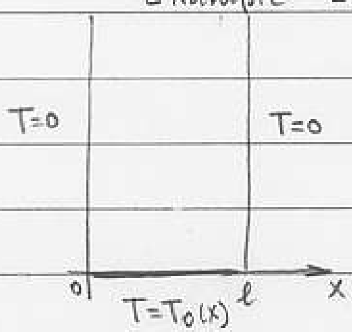
$$X''\cdot Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

Since the first term depends only on x, and the second - only on y, they both has to be constant and opposite for the equation to be valid.

$$\underbrace{\frac{X''}{X}}_{\substack{k^2 \\ \text{or } -k^2}} + \underbrace{\frac{Y''}{Y}}_{\substack{-k^2 \\ k^2}} = 0$$

| | |
|---|---|
| a) $\frac{X''}{X} = -k^2 \qquad \frac{Y''}{Y} = k^2$ | $\frac{X''}{X} = k^2 \qquad \frac{Y''}{Y} = -k^2$ |
| $X'' = -k^2 X \qquad Y'' = k^2 Y$ | $X'' = k^2 X \qquad Y'' = -k^2 Y$ |
| $\{\sin kx, \cos kx\} \qquad \{e^{-ky}, e^{+ky}\}$ | $\{e^{kx}, e^{-kx}\} \qquad \{\cos ky, \sin ky\}$ |
| $X(x) = A\cos kx + B\sin kx \qquad Y(y) = Ce^{ky} + De^{-ky}$ | |
| $X$ is oscillotory, $Y$ is monotonous | $Y$ is oscillotory, $X$ is monotonous |

Example 1: temperature distribution in the semi-infinite slab




$\nabla^2 T(x,y) = 0$

$T(x,y) = X(x)Y(y)$

$X$ must be oscillotory ($X=0$ for $x=0, x=\ell$)

$Y = D e^{-ky}$, since $Y(y\to\infty) = 0$

so $T(x,y) = (A\cos kx + B\sin kx)\, e^{-ky}$

$x=0 \Rightarrow A=0$ ; $X(\ell)=0 \Rightarrow \sin k\ell = 0 \Rightarrow k_n = \frac{\pi n}{\ell} \quad n=0,1,2\ldots$

$$T(x,y) = \sum_{n=0}^{\infty} B_n \sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi n y}{\ell}}$$

last boundary condition: $T(x, y=0) = T_0(x)$

$$T(x,0) = \sum_{n=0}^{\infty} B_n \sin\frac{\pi n x}{\ell} = T_0(x)$$

To find the coefficients, we have to decompos $T_0(x)$ into the sine series $T_0(x) = \sum_{n=0}^{\infty} T_n \sin\frac{\pi n x}{\ell}$

then $B_n = T_n$, and

$$T(x,y) = \sum_{n=0}^{\infty} T_n \sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi n y}{\ell}}$$

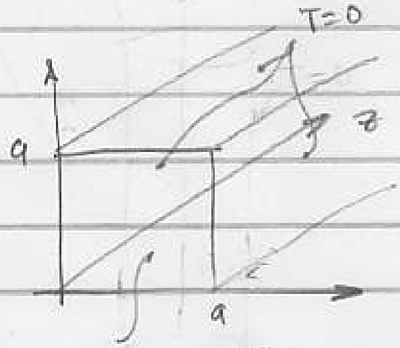
Suppose $T_0(x) = T_0$ – constant temperature

$$T_n = \frac{2}{\ell}\int_0^{\ell} T_0 \cdot \sin\frac{\pi n x}{\ell}\,dx = \frac{2T_0}{\pi n}\left(-\cos\frac{\pi n x}{\ell}\right)\Big|_0^{\ell} = \frac{2T_0}{\pi n}\left(-(-1)^n + 1\right) =$$

$$= \frac{4T_0}{\pi n} \quad n = 1,3,5$$

$$T(x,y) = \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{4T_0}{\pi n} \sin\frac{\pi n x}{\ell}\, e^{-\frac{\pi n y}{\ell}}$$

Example 2: Same situation, but in 3D




$T_0(x,y) = T_0 \frac{x \cdot y}{a^2}$

$$\nabla^2 u = 0 \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(x,y,z) = XYZ$$

$$\underbrace{\frac{X''}{X}}_{C_1} + \underbrace{\frac{Y''}{Y}}_{C_2} + \underbrace{\frac{Z''}{Z}}_{C_3} = 0 \qquad C_1 + C_2 + C_3 = 0$$

$X$ and $y$ are oscillotory $\rightarrow$ so $C_1 < 0 \quad C_2 < 0$

$$X'' = -k_1^2 X \qquad Y'' = -k_2^2 Y$$

since $X(0) = X(a) = 0$ and $Y(0) = Y(a) = 0$

$$X_n = \sin \frac{\pi n x}{a} \qquad Y_m = \sin \frac{\pi m y}{a} \qquad n, m = 0, 1, 2 \ldots$$

Then $C_3 = -C_1 - C_2 = k_1^2 + k_2^2 = \frac{\pi^2 n^2}{a^2} + \frac{\pi^2 m^2}{a^2} = k_z^{n,m}$

$$Z_{n,m}(y) = e^{-k_z^{nm} z}$$

$$T(x,y,z) = \sum_{n,m} A_{nm} \sin \frac{\pi n x}{a} \sin \frac{\pi m x}{a} e^{-k_z^{nm} z}$$

To find the coefficients $A_{nm}$ we have to decompose the function $T_0(x,y) = T_0 \frac{xy}{a^2}$ into a double fouriee series

$$T_0(x,y) = \sum T_{nm} \sin \frac{\pi n x}{a} \sin \frac{\pi m x}{a}$$

To do that we fist do the decomposition in $y$ (treating $x$ as a constant parameter)

$$y = \sum_{m=0}^{\infty} C_m(x) \sin \frac{\pi m y}{a} = T_0 \frac{xy}{a^2} \Rightarrow C_m(x) = \frac{2 T_0 x}{\pi m} (-1)^{m+1}$$

Then we decompos $C_m(x)$ into a Fourier sine series over $X$

$$C_m(x) = \sum_{n=0}^{\infty} \frac{4 T_0}{\pi^2 m n} (-1)^{n+m} \sin \frac{\pi n x}{a}$$

Thus $T_0(x,y) = \sum_{n,m} \underbrace{\frac{4T_0}{\pi^2 n m} (-1)^{n+m}}_{A_{nm}} \sin \frac{\pi n x}{a} \sin \frac{\pi n y}{a}$

$$T(x,y,z) = \sum_{n,m=0}^{\infty} \frac{4 T_0}{\pi^2 n m} (-1)^{n+m} \sin \frac{\pi n x}{a} \sin \frac{\pi n y}{a} e^{-\frac{\pi \sqrt{n^2+m^2}}{a} z}$$