Bessel functions

Equation

$x^2y'' + xy' + (k^2x^2 - p^2)y = 0 \Rightarrow kx \longrightarrow x$ | Solutions: $J_p(kx)$

$x^2y'' + xy' + (x^2 - p^2)y = 0$ | Solutions: $J_p(x)$

In general, for non-integer p there is no power series solutions
However, we can use a <u>generalized</u> power series.

$$y = x^s \sum_{n=0}^{\infty} a_n x^n \quad \text{where "s" is any number}$$

(Example of a generalized power series: $\frac{\sin x}{\sqrt{x}} = x^{-1/2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

We will use <u>Frobenius method</u> to solve the Bessel eqn

$$y = \sum_{n=0}^{\infty} a_n x^{n+s} \qquad y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s-2}$$

$$\underbrace{\sum_{n=0}^{\infty} a_n (n+s)(n+s-1) x^{n+s} + \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}}_{\text{combine}} + \sum_{n=0}^{\infty} a_n x^{n+s+2} - p^2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} - p^2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

We will use the first term $n=0$ to find s

$n=0$: $s^2 - p^2 = 0 \qquad s = \pm p \qquad$ pick $s = p$

Set $a_1 = 0$ (since only a_{n+2} and a_n are connected)

For $n \geq 2$: $\quad a_n[(n+p)^2 - p^2] + a_{n-2} = 0$; $a_n[n^2 + 2np] + a_{n-2} = 0$

$$a_n = -\frac{a_{n-2}}{n(n+2p)}$$

All non-zero terms are even-order: $n = 2k$

Then

$$a_{2k} = -\frac{a_{2k-2}}{2k(2k+2p)} = (-1)^2 \frac{a_{2k-4}}{2k(2k+2p)(2k-2)(2k-2+p)} = (-1)^2 \frac{a_{2k-4}}{2^4 k(k-1)(k+p)(k+p-1)} =$$

$$= (-1)^k \frac{a_0}{2^{2k} k! (k+p)(k+p-1)\ldots\underbrace{(k+p+1-k)}_{p+1}} = (-1)^k a_0 \frac{\Gamma(p)}{2^{2k} k! \Gamma(k+p+1)}$$

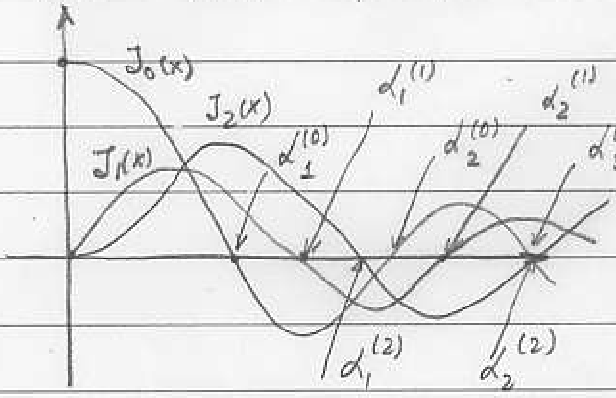
Thus

$$y(x) = x^p \sum_{k=0}^{\infty} (-1)^k a_0 \frac{\Gamma(p+1)}{2^{2k}\, k!\, \Gamma(p+k+1)} x^{2k} = \underbrace{\left[a_0 \Gamma(p+1)\, 2^p\right]}_{=1 \to \text{can choose } a_0} \sum_{k=0}^{\infty} (-1)^k \frac{1}{\underbrace{k!}_{\Gamma(k+1)}\, \Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p}$$

Bessel funcktion

$$\boxed{J_p(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(k+1)\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p}}$$

Bessel functions are somewhat similar to sine and cosine, but their "period" is not stable.



For $x \ll 1$ $k=0$ is the leading term

$$J_p(x) \approx \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p$$

$$J_0 \approx 1, \quad J_1 \approx \frac{1}{2}x, \quad J_2 \approx \frac{1}{8}x^2$$

There are tables of zeros for each Bessel function.

Asymptotic for $x \gg 1, p$

$$J_p(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)$$

for large x the distance b/w zeros is $\approx \pi$.

Recurrence relations

$$x^p J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)} \frac{x^{2k+2p}}{2^{2k+p}}$$

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2p)}{\Gamma(k+1)\Gamma(k+p+1)} \frac{x^{2k+2p-1}}{2^{2k+p}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p)} x^p \frac{x^{2k+p-1}}{2^{2k+p-1}} = x^p J_{p-1}(x)$$

so $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \qquad (1)$$

Similarly $$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x) \qquad (2)$$

Also
$$\left.\begin{aligned} J_{p-1}(x) + J_{p+1}(x) &= \frac{2p}{x} J_p(x) \\ J_{p-1}(x) - J_{p+1}(x) &= 2 J_p'(x) \end{aligned}\right] \quad \text{HW problems}$$

Recurrence relations are very convenient in calculations!

1. $\int x J_0(x)\,dx = \int \frac{d}{dx}[x J_1(x)]\,dx = x J_1(x)$

$\int J_1(x)\,dx = -\int \frac{d}{dx}[J_0(x)]\,dx = -J_0(x)$

Thus, for example $\int_0^\infty J_1(x)\,dx = -J_0(x)\Big|_0^\infty = 1$

$\int_0^\infty J_3(x)\,dx = \int_0^\infty (2J_2'(x) + J_1(x))\,dx = \cancel{2J_2(x)\Big|_0^\infty} + \int_0^\infty J_1(x)\,dx = 1$

and in general $\int_0^\infty J_1(x)\,dx = \int_0^\infty J_3(x)\,dx = \dots \int_0^\infty J_{2n+1}(x)\,dx = 1$

Generation function exists only for integer $p = n$

$$\Phi(x,t) = e^{x/2\,(t - 1/t)} = \sum_{n=-\infty}^{+\infty} t^n J_n(x)$$

(we can use it to find the recurrence relations as well)

Important physics application → phase modulation

$\Phi(x,t) = e^{x/2(t - 1/t)} \Rightarrow t = e^{i\varphi}$ $t = e^{-i\varphi}$, $\frac{x}{2}(t - \frac{1}{t}) = ix\sin\varphi$

$$\Phi(\varphi,t) = e^{ix\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(x)\,e^{in\varphi}$$

For the electromagnetic wave $E(z,t) = E_0\,e^{ikz - i\omega t + i\varphi}$

if the phase is modulated $\varphi(t) = \varepsilon_m \sin\omega_m t$ $(\omega_m \ll \omega)$

$$e^{i\varphi(t)} = e^{i\varepsilon_m \sin\omega_m t} = \sum_{n=-\infty}^{+\infty} J_n(\varepsilon_m)\,e^{i\omega_m t\cdot n}$$

And the resulting field is

$$E(z,t) = E_0 \sum_{n=-\infty}^{+\infty} J_n(\varepsilon_m)\,e^{ikz - i\omega t + in\omega_m t} =$$

$$= \sum_{n=-\infty}^{+\infty} [E_0 J_n(\varepsilon_m)]\,e^{ikzt - i(\omega - n\omega_m)t}$$ ← frequency comb!

Coming back to math: using the generation function one can derive the integral representation of the $J_n(x)$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\varphi - x\sin\varphi)\,d\varphi$$

Other Bessel functions.

If you remember, for the Frobenius method we picked the solution with $p = s$. Generally $p = -s$ gives the second independent solution for the Bessel equation (Except for $p = n$, then $J_{-n}(x) = (-1)^n J_n(x)$)

$$J_{-p}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(k+1)\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}$$

In general, $J_p(x)$ are called the Bessel functions of the first kind

It's "complimentary" function : Neumann or Weber functions or the Bessel functions of the second kind

$$N_p(x) = Y_p(x) = \frac{\cos \pi p \, J_p(x) - J_{-p}(x)}{\sin \pi p}$$

$$N_n(x) = \lim_{p \to n} N_p(x)$$

Loose analogy basis $(\cos x, \sin x) \longleftrightarrow$ basis (e^{ix}, e^{-ix})

basis $(J_p(x), N_p(x)) \longleftrightarrow$ basis $(H_p^{(1)}(x), H_p^{(2)}(x))$

Hankel functions (Bessel functions of the third type)

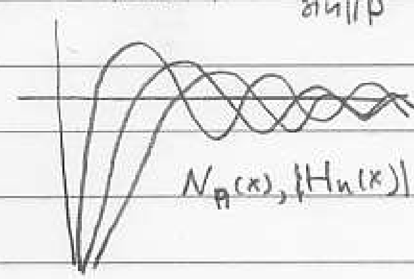
$$H_p^{(1,2)}(x) = J_p(x) \pm i N_p(x) = J_p(x) \pm i \, \frac{\cos \pi p \, J_p(x) - J_{-p}(x)}{\sin \pi p}$$

$$= \frac{J_p(x)\left[\sin \pi p \pm i \cos \pi p\right] - i J_{-p}(x)}{\sin \pi p} = \frac{J_{-p}(x) \mp e^{ip\pi} J_p(x)}{i \sin \pi p}$$

Assymptotics: $x \ll 1$ $\quad J_{-p}(x) \approx \frac{1}{\Gamma(-p+1)} \left(\frac{x}{2}\right)^{-p} \Rightarrow \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$

$$J_{-p}(x) \approx \frac{\Gamma(p) \cdot \sin \pi p}{\pi} \left(\frac{x}{2}\right)^{-p}$$

$$N_p(x)\Big|_{x \ll 1} \approx -\frac{\pi}{\sin \pi p} J_{-p}(x) \approx -\frac{\Gamma(p)}{\pi} \left(\frac{x}{2}\right)^{-p}$$



$N_n(x), |H_n(x)|$

For large $x \gg 1, p$

$$N_p(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi p}{2} - \frac{\pi}{4}\right)$$

$$H_p^{(1,2)}(x) \approx \sqrt{\frac{2}{\pi x}} \, e^{\pm i\left(x - \frac{\pi p}{2} - \frac{\pi}{4}\right)}$$

Modified Bessel functions (of the first and the second kind)

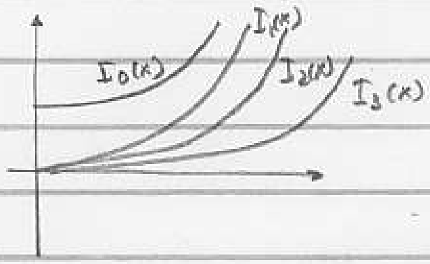
Analogy $(\sin x, \cos x) \longleftrightarrow (\sinh x, \cosh x)$

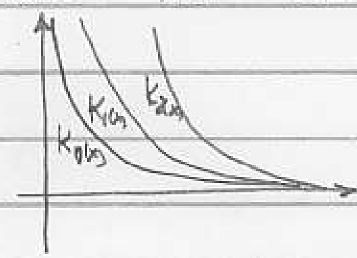
$$I_p(x) = i^{-p} J_p(ix) \qquad K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix)$$

If x is real, I_p is real!

$$I_p(x) = \cancel{i^{-p}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p} \cancel{i^{2k+p}} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}$$

Modified Bessel functions are not oscillatory





Spherical Bessel functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) \qquad n - \text{integer}$$

$$y_n(x) = n_n(x) = \sqrt{\frac{\pi}{2x}}\, N_{n+1/2}(x)$$

$$h_n^{(1,2)}(x) = j_n(x) \pm i y_n(x)$$

Spherical Bessel functions often appear in the solutions of 3D Helmholtz equation

$$j_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \qquad y_n(x) = -x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$$

Recurrence relations

$$\frac{d}{dx}\left[x^{n+1} j_n(x)\right] = x^{n+1} j_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n} j_n(x)\right] = -x^{-n} j_{n+1}(x)$$