Solutions of differential equations.

We will consider a special class of differential equs, that are called Sturm - Liouville problem

$$\frac{d}{dx}\left(f(x)\frac{dy}{dx}\right) - g(x)y + \lambda w(x) y = 0$$

Such equations often appear in various physics problems, for example in solutions of Schrödinger equation:

$$\hat{H}\psi(x) = E\psi(x)$$

One has to solve this equation for a particular problem (i.e for different forms of $\hat{H} = \hat{K} + \hat{V}$), find all real eigen values E_n of energy, and the corresponding eigen functions ψ_n.

We also usually search for a solution at a specific region $a \le x \le b$, and our solution must obey some imposed boundary conditions, for example:

Dirichlet conditions: $y(a) = y(b) = 0$

Neumann conditions: $y'(a) = y'(b) = 0$

Let's show that solutions $\{y_n\}$ of the Sturm - Liouville problem corresponding to different eigen values λ_n are orthogonal.

But first we need to define the product of two functions:

$$\langle u(x) | v(x) \rangle = \int_a^b u^*(x)\, v(x) \cdot w(x)\, dx$$

$w(x)$ is a weight function

(It will determine the assimptotic behavior of all solutions)

So let's check that $\langle y_n | y_m \rangle = 0$ if $\lambda_n \neq \lambda_m$
(and we will assume all $\{y_m\}$ are real)

$$y_m \times \frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) - g(x)\,y_n + \lambda_n w(x)\,y_n(x) = 0$$

$$-\quad y_n \times \frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right) - g(x)\,y_m + \lambda_m w(x)\,y_m(x) = 0$$

(subtract and integrate)

$$\underbrace{\int_a^b \left[y_m \frac{d}{dx}\left(f(x)\frac{dy_n}{dx}\right) - y_n \frac{d}{dx}\left(f(x)\frac{dy_m}{dx}\right)\right] dx}\; + (\lambda_n - \lambda_m)\int_a^b w(x)\,y_n(x)\,y_m(x)\,dx = 0$$

$$= \underbrace{\cancel{\left[\left(y_m f(x)\frac{dy_n}{dx}\right)\Big|_a^b - \left(y_n f(x)\frac{dy_m}{dx}\right)\Big|_a^b\right]}}_{=0 \text{ because of the boundary conditions}} - \int_a^b \left(f(x)\frac{dy_n}{dx}\frac{dy_m}{dx}\,dx - f(x)\frac{dy_m}{dx}\frac{dy_n}{dx}\right)dx = 0$$

Thus $(\lambda_n - \lambda_m)\int_a^b w(x)\,y_n(x)\,y_m(x) = 0$, and if $\lambda_n \neq \lambda_m$
solutions $y_n(x)$ and $y_m(x)$ are <u>orthogonal</u> $\langle y_n | y_m \rangle = 0$

It is also possible to show that solutions $\{y_n\}$ form a <u>complete</u> set of functions, i.e. any function $F(x)$ can be decomposed into a sum of $\{y_n\}$

$$F(x) = \sum_{n=0}^{\infty} a_n\, y_n(x)$$

Using the orthogonality:

$$\int_a^b F(x)\,y_k(x)\,w(x)\,dx = \sum_{n=0}^{\infty} a_n \int_a^b y_n(x)\,y_k(x)\,w(x)\,dx = a_k \int_a^b y_k^2\, w(x)\,dx$$

$$a_k = \frac{\int_a^b F(x)\,y_k(x)\,w(x)\,dx}{\int_a^b y_k^2\, w(x)\,dx}$$

Fourier series is an example of Sturm - Liouville problem

$\frac{d^2}{dx^2} y + k^2 y = 0 \quad \Rightarrow \quad w(x) = 1, \quad f(x) = 1, \quad g(x) = 0, \quad \lambda = k^2$

With Dirichlet boundary conditions

$y(L) = y(-L) = 0 \qquad y_n = \sin \frac{\pi n x}{L}, \ k_n = \frac{\pi n}{L}$

With Neumann boundary conditions

$y'(L) = y'(-L) = 0 \qquad y_n = \cos \frac{\pi n x}{L}, \ k_n = \frac{\pi n}{L}$

Another important example: <u>Legandre polynomials</u>

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \ell(\ell+1)y = 0 \quad \text{or} \quad (1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

We will search for solutions in forms of power series that are finite at the interval $\underline{-1 \le x \le 1}$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$
$$y'(x) = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$
$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + \ldots$$

Substitute y, y' and y'' into the Legandre eqn

$$(1-x^2)(2a_2 + 6a_3 x + \ldots) - 2x(a_1 + 2a_2 x + 3a_3 x^2) + \ell(\ell+1)(a_0 + a_1 x + a_2 x^2 + \ldots) = 0$$

and then collect the coefficient for each power of x and make sure each of them is zero.

$x^0: \ 2a_2 + \ell(\ell+1)a_0 = 0 \qquad a_2 = -\frac{\ell(\ell+1)}{2} a_0$

$x^1: \ 6a_3 - 2a_1 + \ell(\ell+1)a_1 = 0 \qquad a_3 = \frac{\ell^2+\ell-2}{6} a_1 = \frac{(\ell+2)(\ell-1)}{6}$

For an arbitrary x^n

$$(1-x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} a_n \cdot n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^n - 2\sum_{n=1}^{\infty} a_n n x^n + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Big\downarrow\; n \to n+2$$

$$\sum a_{n+2}(n+2)(n+1)x^n - \ldots$$

$$a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n \cdot n + l(l+1)a_n = 0$$

$$a_{n+2} = \frac{[n(n+1) - l(l+1)]}{(n+1)(n+2)}a_n = -\frac{(l+n+1)(l-n)}{(n+1)(n+2)}a_n =$$

$$= \frac{(l+n-1)(l-n)}{(n+1)(n+2)}\,\frac{(l+n-3)(l-n+2)}{n(n-1)}\,a_{n-2} = \ldots$$

Thus the general solution of the Legendre eqn is

$$y = a_0\left(1 - \frac{l(l+1)}{2}x^2 + \frac{(l+3)(l+1)l(l-2)}{4\cdot 3\cdot 2\cdot 1 \;\; 4!}x^4 - \ldots\right) +$$

$$+ a_1\left(x - \frac{(l+2)(l-1)}{3!}x^3 + \frac{(l+4)(l+2)(l-1)(l-3)}{5!}x^5 - \ldots\right)$$

This infinite series converges for $|x| < 1$.
A special case is l = integer number.
Then one series will become a finite polynomial, and the other will still be an infinite series diverging at $x = \pm 1$.
Thus to have a solution for $|x| \le 1$, we have to keep the polynomial solution, and set the other series to zero.
These are <u>Legendre polynomials</u>

Traditional normalization of Legendre polynomials

$P_\ell(1) = 1$ $\qquad P_0(x) = 1$

$P_1(x) = x$

$\ell = 2: \quad P_2 = a_0\left(1 - \frac{\cancel{2}\cdot 3}{2}x^2\right) \Rightarrow -2a_0 = 1 \quad a_0 = -\frac{1}{2}$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

Legendre polynomials are eigen function of the parity operator

$P_\ell(x) = P_\ell(-x)$ for even ℓ, and $P_\ell(x) = -P_\ell(-x)$ for odd ℓ

$P_\ell(x) = (-1)^\ell P_\ell(-x)$

Rodriges formula to derive Legendre polynomials

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

Before prooving this, let's first notice that if $v(x) = (x^2-1)^\ell$

$$(x^2-1)\frac{dv}{dx} = (x^2-1)\,2\ell x\,(x^2-1)^{\ell-1} = 2\ell x \cdot v(x)$$

We will aslo use the Leibniz's differentiation rule:

$$\frac{d^n}{dx^n}(f(x)g(x)) = \frac{d^{n-1}}{dx^{n-1}}\left(f(x)g'(x) + g(x)f'(x)\right) = \frac{d^{n-2}}{dx^{n-2}}(fg'' + 2f'g' + f''g) =$$

$$= \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)$$

That means that

$$\frac{d^{\ell+1}}{dx^{\ell+1}}\left[(x^2-1)\frac{dv}{dx}\right] = (x^2-1)\frac{d^{\ell+2}}{dx^{\ell+2}}v + 2x(\ell+1)\frac{d^{\ell+1}}{dx^{\ell+1}}v + \frac{\ell(\ell+1)}{2}\cdot 2\,\frac{d^\ell}{dx^\ell}v$$

on the other hand //

$$\frac{d^{\ell+1}}{dx^{\ell+1}}(2\ell x\cdot v) = 2\ell x\frac{d^{\ell+1}}{dx^{\ell+1}}v + 2\ell(\ell+1)\frac{d^\ell}{dx^\ell}v$$

$$(x^2-1)\frac{d^2}{dx^2}\left(\frac{d^\ell}{dx^\ell}\cdot v\right) + 2x\frac{d}{dx}\left(\frac{d^\ell}{dx^\ell}v\right) - \ell(\ell+1)\frac{d^\ell}{dx^\ell}v = 0$$

$$(1-x^2)\frac{d^2}{dx^2}v^{(\ell)} - 2x\frac{d}{dx}v^{(\ell)} + \ell(\ell+1)v^{(\ell)} = 0$$

Thus $y = A \cdot v^{(\ell)}$ is a solution of Legendre eqn.
To make it the Legendre polynomials one needs
to take care of normalization

$$P_\ell(1) = 1 = A \cdot \frac{d^\ell}{dx^\ell} (x^2-1)^\ell \Big|_{x=1} = A \cdot \frac{d^\ell}{dx^\ell} (x-1)^\ell (x+1)^\ell \Big|_{x=1} =$$

Any terms with $(x-1)$ will be zero at $x=1$.
So the only non-zero term is when $(x-1)^\ell$ is differentiaten ℓ times, and $(x+1)^\ell$ term is left as is.

$$= A \cdot \ell! \, (x+1)^\ell \Big|_{x=1} = A \cdot \ell! \cdot 2^\ell \quad \Rightarrow \quad A = \frac{1}{2^\ell \ell!}$$

$$\boxed{P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell}$$