Methods to diagnose the quantum states

1. Photon counting (decomposition into the number state basis)
   reliable, but complicated by the lack of highly-efficient number-resolving detectors
   $\hat{\rho} = \sum_{nm} \rho_{nm} |m\rangle\langle n|$
2. Homodine quadrature measurements
   "easy", but does not provide complete information about the states

3. Quantum tomography → reconstruction of a Wigner function of a quantum state

Wigner function is ~~an~~ ~~an~~ a quantum analogue of a probability distribution function
However, Wigner function is a quazi-probability function, and can be negative.

~~Number state~~ Coherent-state basis

$$\hat{\rho} = \iint \langle \alpha' | \rho | \alpha'' \rangle |\alpha'\rangle\langle\alpha''| \frac{d^2\alpha' d^2\alpha''}{\pi^2}$$

$\Downarrow$ want to use

$$\hat{\rho} = \int P(\alpha) \, |\alpha\rangle\langle\alpha| \, d^2\alpha$$

$P(\alpha)$ – Glauber – Sudarshan P-function
For classical states $P(\alpha) > 0$ or no more singular than a δ-function
If $P(\alpha)$ is negative or highly singular ($\sim \frac{d\delta}{\partial x}$ or higher derivatives) – the state is non-classical

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int e^{|u|^2} \langle -u|\rho|u\rangle e^{u^*\alpha - u\alpha^*} d^2u$$

A coherent state $-\ \hat{\rho} = |\beta\rangle\langle\beta|$

$$P(\alpha) = \delta^2(\alpha - \beta)$$

A number state $\hat{\rho} = |n\rangle\langle n|$

$$P(\alpha) = \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}} \delta^{(2)}(\alpha)$$

Another handy representation : Q-function

$Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle/\pi$ always positive

Both representations can be ~~handy~~ ~~to~~ used to calculate the expectation values of an operator

$\hat{B} = \int B_P(\alpha, \alpha^*)\, |\alpha\rangle\langle\alpha|\, d^2\alpha$ P-representation

$B_Q(\alpha, \alpha^*) = \langle\alpha|\hat{B}|\alpha\rangle$ Q-representation

$$\langle\hat{B}\rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \int B_P(\alpha, \alpha^*) \underbrace{\langle\alpha|\hat{\rho}|\alpha\rangle}_{Q(\alpha)} d^2\alpha$$

or

$$\langle\hat{B}\rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \int P(\alpha)\, B_Q(\alpha, \alpha^*)\, d^2\alpha$$

A Wigner function : phase-space quazi-probability distribution

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \langle q + \tfrac{1}{2}x|\, \hat{\rho}\, |q - \tfrac{1}{2}x\rangle\, e^{ipx/\hbar}\, dx$$

originally $\hat{q}$ and $\hat{p}$ are the position and momentum operators

if $\hat{\rho} = |\psi_q\rangle\langle\psi_q| = |\psi_p\rangle\langle\psi_p|$

$$\int_{-\infty}^{+\infty} W(q,p)\, dp = |\psi_q(q)|^2$$ probability density distribution for q

$$\int_{-\infty}^{+\infty} W(q,p)\, dq = |\psi_p(p)|^2$$ probability density distribution in p-space

In quantum optics the variables p and q are related to quadrature operators X_1 and X_2 (or X_θ and $X_{\pi/2+\theta}$)
For example, to find a propability distribution for ~~values~~ possible values X_θ :

$$pr(X,\theta) = \int_{-\infty}^{+\infty} W(q\cos\theta - p\sin\theta,\ q\sin\theta + p\cos\theta)\, dp$$

~~fo~~ for a given value of θ.

This gives an opportunity to reconstruct the Wigner function from the series of quadrature value measurements for several values of the lockal oscillator (i.e. mapping the spread of X_θ for each value of θ), and then performing a mathematical inversion operation, called inverse Radon transformation

$$W(q,p) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{0}^{\pi} \int_{-\infty}^{+\infty} \underbrace{pr(x,\theta)}_{\text{experimentally measurable}} |\xi|\, e^{i\xi(q\cos\theta + p\sin\theta - x)}\, dx\, d\theta\, d\xi$$

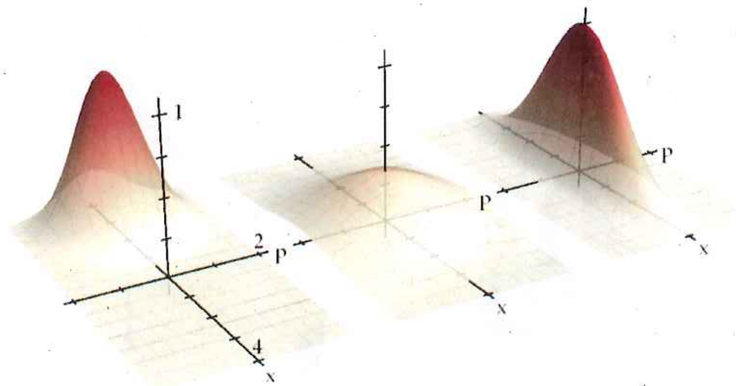


FIGURE 2.1: Wigner functions of, from left to right, a coherent state with amplitude $\alpha = -2$, a thermal state with mean photon number $n_{\text{th}} = 1$, and a squeezed vacuum state with squeezing level $r = \ln 2$ (squeezed variance $1/8$). The function values have been multiplied by π, which makes 1 and -1 the maximum and minimum values attainable. We will be doing this scaling in all plots and graphs throughout the thesis.

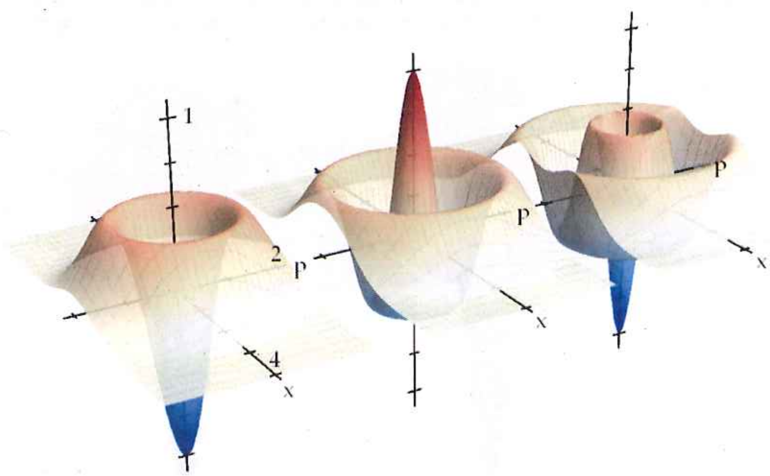


FIGURE 2.2: $\pi W(x, p)$ for the 1-, 2-, and 3-photon Fock states, from left to right. Notice their alternating negative and positive values in the origin.

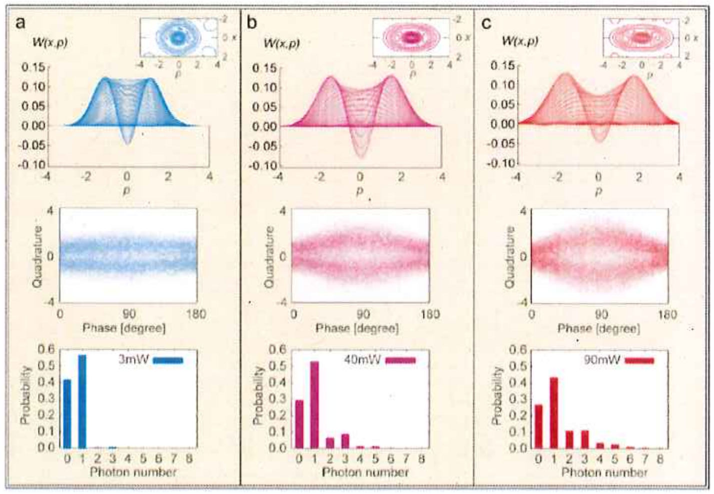


Fig. 2. Experimental Wigner functions (top panels) constructed from raw data without any correction of measurement imperfections, in the case of 5% splitting ratio. a : The single-photon state generated by −0.7dB initial squeezed vacuum. b and c : Schrödinger kittens generated by −2.6dB and −3.7dB initial squeezed vacua, respectively. The values of the Wigner function at the origin are a : $W(0,0) = -0.049$, b : $W(0,0) = -0.083$, and c : $W(0,0) = -0.048$. The insets in top panels are the contours of the Wigner functions. The middle panels are quadrature distributions obtained by homodyne detection. The bottom panels are photon-number distributions obtained by the iterative maximum-likelihood estimation.

How to generate 8th Shrodinger cat state

Even cat $|\psi_e\rangle = N_e\,(|\alpha\rangle + |-\alpha\rangle)$

$|\psi_o\rangle = N_o\,(|\alpha\rangle - |-\alpha\rangle)$

$|\psi_{YS}\rangle = \frac{1}{\sqrt{2}}\,(|\alpha\rangle + i|-\alpha\rangle)$

$|\psi_e\rangle$ and $|\psi_o\rangle$ cannot be generated using a unitary transformation, but the Yurke-Stoler state can.

$\hat{H}_{int} \propto (\hat{a}^+)^2, (\hat{a}^2)$ $\longrightarrow$ squeezed vaccum is generated (not a cat state)

$\hat{H}_{int} = \hbar K\,(\hat{a}^+\hat{a})^2 = \hbar K\,\hat{n}^2$

$|\psi(t)\rangle = e^{-i\hat{H}_{int}t/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-ikn^2t}|n\rangle$

a periodic function with a period $T = \frac{2\pi}{K}$

If $t = \pi/K$ $\quad e^{-ikn^2t} = e^{-i\pi n^2} = (-1)^{n^2} = (-1)^n$

$|\psi(\pi/K)\rangle = |-\alpha\rangle$

Fot $t = \pi/2k$: $|\psi(\pi/2k)\rangle = \frac{1}{\sqrt{2}}\,e^{-i\pi/4}\,(|\alpha\rangle + i|-\alpha\rangle)$

Possible, but unclear how to realize such hamiltonian

Conditional measurements

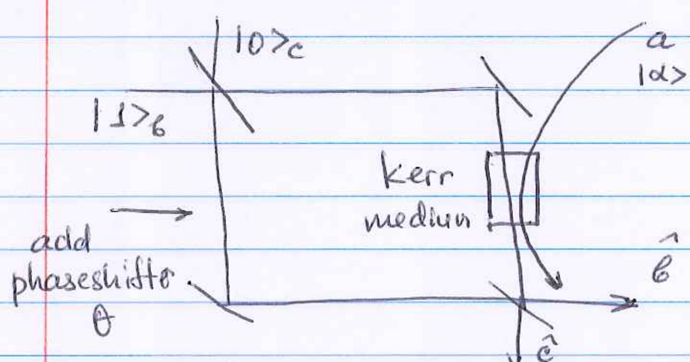
Necessary component: cross - Kerr interaction

$$\hat{H}_{ck} = \hbar K \, \hat{a}^{+}\hat{a} \, \hat{b}^{+}\hat{b} = \hbar K \, \hat{n}_a \hat{n}_b$$

The number of photons in one field affects the state of the other (and vice versa)

$$\hat{U}_{ck} = e^{-itK \hat{a}^{+}\hat{a} \hat{b}^{+}\hat{b}}$$

Mach - zender interferometer



Input states: $|\alpha\rangle_a |1\rangle_b |0\rangle_c$

Inside interferometer / before Kerr medium

$$|\alpha\rangle_a \frac{1}{\sqrt{2}} \left(|1\rangle_b |0\rangle + i |0\rangle |1\rangle \right)$$

Interaction with the Kerr medium

$$e^{-itK \hat{a}^{+}\hat{a} \hat{b}^{+}\hat{b}} |\alpha\rangle|1\rangle = \underbrace{e^{-itK\hat{a}^{+}\hat{a}}}_{\text{phase - shifter}} |\alpha\rangle|1\rangle = |\alpha e^{-iKt}\rangle|1\rangle$$

$$e^{-itK \hat{a}^{+}\hat{a} \hat{b}^{+}\hat{b}} |\alpha\rangle|0\rangle = |\alpha\rangle|0\rangle$$

Before the second beamsplitter

$$\frac{1}{\sqrt{2}} \left(|\alpha e^{-iKt}\rangle |1\rangle|0\rangle + i \, |\alpha\rangle|0\rangle|1\rangle \right)$$

$e^{i\theta}$

Let's adjust Kt such that $Kt = \pi$ $\left(t = \frac{l \cdot n}{c}\right)$

then the state before the beamsplitter

$$\frac{1}{\sqrt{2}}\left(|-\alpha\rangle|1\rangle|0\rangle + i e^{i\theta}|\alpha\rangle|0\rangle|1\rangle\right)$$

After the beamsplitter

$$|out\rangle = \frac{1}{2}\left[\,|-\alpha\rangle\left(|1\rangle|0\rangle + i|0\rangle|1\rangle\right) + i e^{i\theta}|\alpha\rangle\left(|0\rangle|1\rangle + i|1\rangle|0\rangle\right)\right] =$$

$$= \frac{1}{2}\left[\left(|-\alpha\rangle - e^{i\theta}|\alpha\rangle\right)|1\rangle|0\rangle + i\left(|-\alpha\rangle + e^{i\theta}|\alpha\rangle\right)|0\rangle|1\rangle\right]$$

If we want to generate an even or odd cat state $\theta = \pi$

$$|out\rangle = \frac{1}{2}\left[\left(|-\alpha\rangle + |\alpha\rangle\right)|1\rangle|0\rangle - i\left(|\alpha\rangle - |-\alpha\rangle\right)|0\rangle|1\rangle\right]$$

if the detector "B" clicks $\rightarrow$ we have an even cat state in "a" channel
if the detector "C" clicks $\rightarrow$ we have an odd cat state in "a" channel

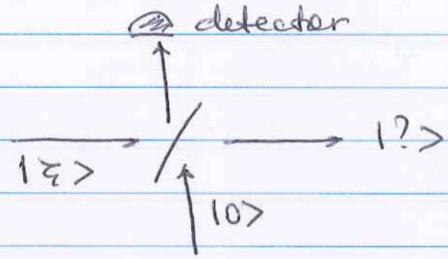
$\theta = \pi/2$ $\longrightarrow$ outputs are two Yurke-Stoler states $\left(|\alpha\rangle \pm i|-\alpha\rangle\right)$

Another method – photon subtraction

Squeezed vacuum state $\xi = re^{i\chi}$

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left(\frac{e^{i\chi} \tanh r}{2} \right)^n |2n\rangle$$

only even number of photons



We assume to use a perfect photon number resolving detector

if even number of photons is detected (subtracted) $\rightarrow$ even number is transmitted

$n=2$: $(\hat{a})^2 |\xi\rangle \rightarrow$ state similar to $|\psi_e\rangle$ for small $|\alpha\rangle$

if odd number of photons is detected $\rightarrow$ $\rightarrow$ odd number of photons is transmitted

$n=1$ $\hat{a}|\xi\rangle \longrightarrow$ similar to $|\psi_o\rangle$ for small $|\alpha\rangle$

Schrodinger kittens!