Entanglement

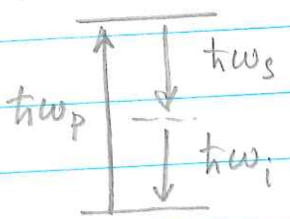
Two systems are placed in a quantum state that cannot be factorized into a product of individual systems.

Example: spontaneous emission

$$|\psi\rangle = e^{-\Gamma/2\, t}|b,0\rangle + \sum_i |a,1_i\rangle c$$

For $t \lesssim 1/\Gamma$ the states of an atom and of a spontaneous photons are entangled (i.e. making a measurement on the atomic state will provide information about photons)

Most common sources of entangled photons – parametric down conversion



1 pump photon (2ω) $\longrightarrow$

$\longrightarrow$ 1 signal + 1 idler photons (ω)

Nonlinear interaction

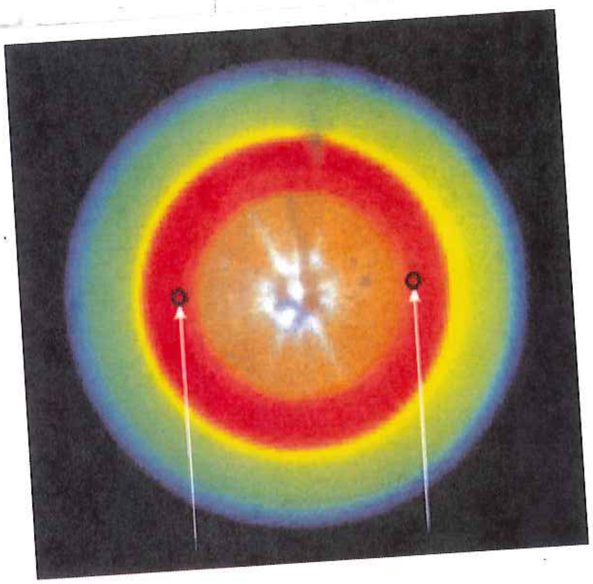
$$\hat{H}_{int} = \chi^{(2)}\, \hat{a}_p \hat{a}_s^{+} \hat{a}_i^{+} + c.c$$

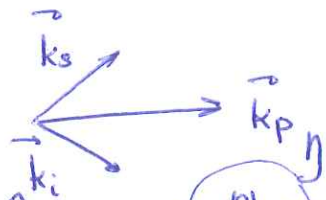
$$\simeq \chi^{(2)} d_p\, \hat{a}_s^{+} \hat{a}_i^{+}$$ (for strong coherent pump laser

Energy conservation: $\hbar\omega_p = \hbar\omega_s + \hbar\omega_i$

Momentum conservation $\hbar\vec{k}_p = \hbar\vec{k}_i + \hbar\vec{k}_s$

$\rightarrow$ Phase-matching conditions





$$\hat H_{int} = \chi^{(2)} \alpha_p \, \hat a_s^+ \hat a_i^+ + h.c.$$

Initial state for signal & idler

$$|\psi_0\rangle = |0\rangle_s |0\rangle_i$$

$$|\psi(t)\rangle = e^{-i\hat H_{int} \cdot t/\hbar} |\psi(t=0)\rangle \approx$$

$$\approx \left(1 - \frac{i\chi\alpha_p}{\hbar} t \, \hat a_s^+ \hat a_i^+ + \frac{1}{2}\left(-\frac{i\chi\alpha_p}{\hbar}\right)^2 t^2 (\hat a_s^+)^2 (\hat a_i^+)^2 + \dots\right) |0\rangle_s |0_i\rangle =$$

$$\chi^{(2)} \frac{\alpha_p}{\hbar} t \ll 1$$

$$\approx |0\rangle_s |0_i\rangle + \overset{\mu}{\left(\frac{i\chi^{(2)}\alpha_p}{\hbar} t\right)} |1\rangle_s |1\rangle_i + \dots$$

$$= |0\rangle_s |0_i\rangle - \mu \, \underbrace{|1\rangle_s |1\rangle_i}_{\text{correlated photon pair}}$$

correlated ~~pai~~ photon pair

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For weak pumping

$$|\psi(t)\rangle = e^{-i\hat{H}_{int} t/\hbar}\, |\psi(0)\rangle \qquad |\psi(0)\rangle = |0\rangle_s |0\rangle_i$$

$$|\psi(t)\rangle \approx \left(1 - i\hat{H}_{int} t/\hbar + \frac{1}{2}\left(-it\hat{H}_{int}/\hbar\right)^2 + \dots\right) |\psi(0)\rangle =$$

$\mu \ll 1$

$$= \left(1 - \boxed{\frac{i\chi^{(2)}\alpha_p}{\hbar}}\, t\, \hat{a}_s^+ \hat{a}_i^+ + \dots\right)|0\rangle_s |0_i\rangle \approx |0\rangle_s|0_i\rangle - i\mu|1\rangle_s|1_i\rangle$$

Source of correlated photons, but not an entangled state

However, one can use this process to generate entanglement, if we consider two possible polarizations

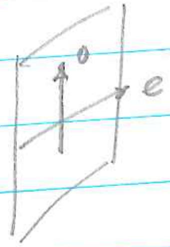
Because of the phase-matching conditions

$$\hbar \frac{n_p \omega_p}{c} \vec{e}_p = \hbar \frac{n_i \omega_i}{c} \vec{e}_i + \hbar \frac{n_s \cdot \omega_s}{c} \cdot \vec{e}_s$$

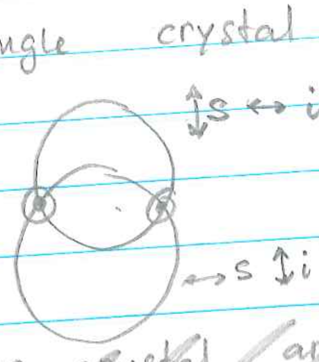
if the refractive indices n_i and n_s are different for different polarizations, they will phase-match at different angles.

Birefringent nonlinear crystals

$n_e \neq n_o$



single crystal



Type II down-conversion

~~Alternatively two crystal arrangement~~

~~two overlapping cones with orthogonal polarizations~~

$$\hat{H}^{(2)}_{int} = \hbar g\left(\hat{a}^{+}_{V_s}\hat{a}_{H_i} + \hat{a}^{+}_{H_s}\hat{a}^{+}_{V_i}\right) + H.c.$$

$$|\psi^{(2)}(t)\rangle \approx |0\rangle_{Vs}|0\rangle_{Hs}|0\rangle_{Vi}|0\rangle_{Hi} -$$

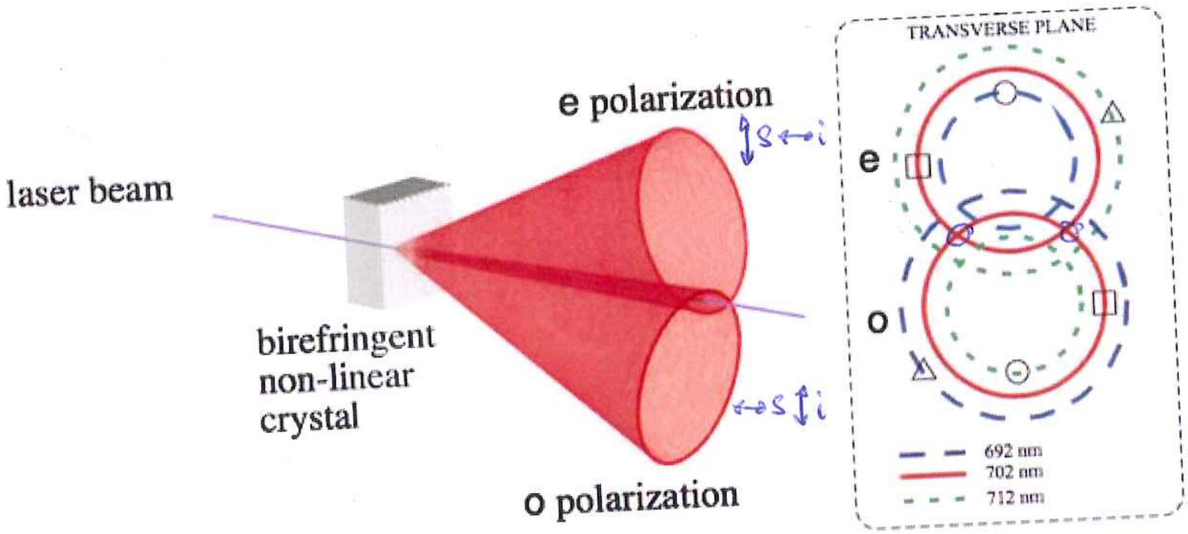
$$- i\mu\,\frac{1}{\sqrt{2}}\left(\underbrace{|1\rangle_{Vs}|0\rangle_{Hs}}_{|V\rangle_s}\underbrace{|0\rangle_{Vi}|1\rangle_{Hi}}_{|H\rangle_i} + \underbrace{|0\rangle_{Vs}|1\rangle_{Hs}}_{|H\rangle_s}\underbrace{|1\rangle_{Vi}|1\rangle_{Vs}}_{|V\rangle_i}\right)$$

$$|\psi^{(2)}(t)\rangle \approx |0\rangle_s|0\rangle_i - \frac{i\mu}{\sqrt{2}}\Big(\underbrace{|V\rangle_s|H\rangle_i + |H\rangle_s|V\rangle_i}_{\text{polarization-entangled two-photon state}}\Big)$$

One of Bell states

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|V\rangle_s|H\rangle_i \pm |H_s\rangle|V_i\rangle\right)$$

$$|\varphi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_s|H\rangle_i \pm |V\rangle_s|V\rangle_i\right)$$



$$\hat{H}_{int} = \hbar\eta\,(\hat{a}^{+}_{Vs}\,\hat{a}^{+}_{Hi} + \hat{a}^{+}_{Hs}\,\hat{a}^{+}_{Vi}) + H.c$$

$$|\psi_0\rangle = |0\rangle_{sV}\,|0\rangle_{sH}\,|0\rangle_{iV}\,|0\rangle_{iH}$$

$$|\psi(t)\rangle = |0\rangle_{sV}|0\rangle_{sH}|0_{iV}\rangle|0_{iH}\rangle + \mu\frac{1}{\sqrt{2}}\left(|1\rangle_{sV}|0\rangle_{sH}|0\rangle_{iV}|1\rangle_{iH} + |0\rangle_{sV}|1\rangle_{sH}|1\rangle_{iV}|0\rangle_{iH}\right)$$

$|0\rangle_{V}\,|1\rangle_{H} = |H\rangle$

$|1\rangle_{V}\,|0\rangle_{H} = |V\rangle$

$$|\psi(t)\rangle = |0\rangle_{vac} + \frac{\mu}{\sqrt{2}}\underbrace{\left(|V\rangle_s|H\rangle_i + |H\rangle_s|V_i\rangle\right)}_{\text{entangled state}}$$

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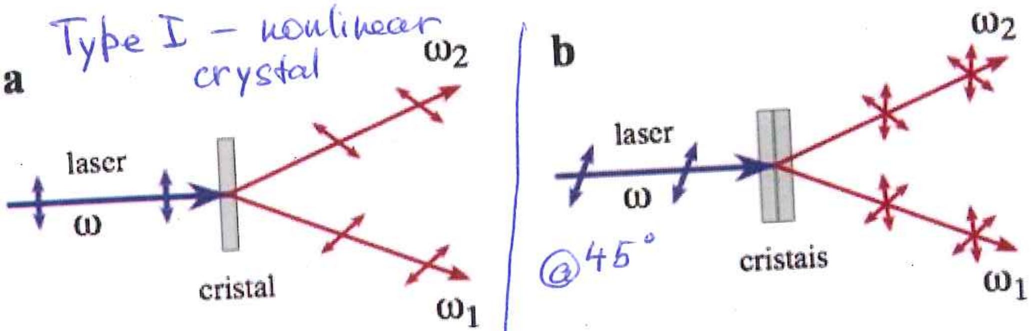
Type I – nonlinear crystal



@45°

$$|H\rangle_p \rightarrow |V\rangle_s |V\rangle_i$$

$$\hat{H}_{int} = \hbar \eta\, a^\dagger_{Vs} a^\dagger_{Vi} + h.c.$$

$$\hat{H}_{int} = \hbar \eta \left(\hat{a}^\dagger_{Vs} \hat{a}^\dagger_{Vi} + \hat{a}^\dagger_{Hs} \hat{a}^\dagger_{Hi}\right) + h.c.$$

$$|\psi(t)\rangle = |vac\rangle + \frac{1}{\sqrt{2}} \mu \left(|V\rangle_s |V\rangle_I + |H\rangle_s |H\rangle_I \right)$$

entangled state

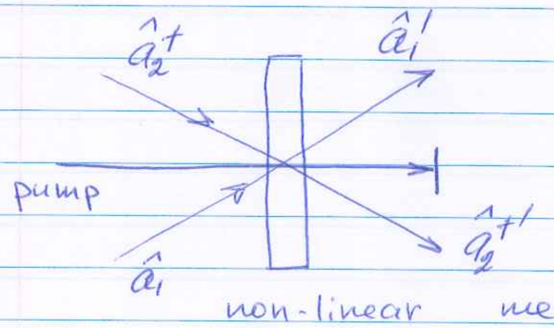
Bell states

eigen-basis for two-particles entangle states are

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}} \left(|V\rangle_s |H\rangle_I \pm |H\rangle_s |V\rangle_I \right)$$

$$|\varphi^\pm\rangle = \frac{1}{\sqrt{2}} \left(|V\rangle_s |V\rangle_I \pm |H\rangle_s |H\rangle_I \right)$$

Optical parametric amplification



$\hat{a}_1, \hat{a}_2^+$ – input modes (can be a vacuum states)

$\hat{a}_1'\ \hat{a}_2^{+\prime}$ – output modes

Typical Hamiltonian $\hat{H}_{OPA} \propto \chi^{(3)}\ \hat{a}_p \hat{a}_1^+ \hat{a}_2^+ + \hat{a}_p^+ \hat{a}_1 \hat{a}_2$

replacing $\hat{a}_p \rightarrow \alpha_p$ (strong coherent pump)

Solution: $\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2^{+\prime} \end{pmatrix} = A \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^+ \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{\text{Bogoliubov transformation}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^+ \end{pmatrix}$

Same ~~behav~~ behaviour ~~ca~~ can occure in case of other non-linear processes (i.e. four-wave mixing)

To preserve the commutation relationships for $\hat{a}_1$ and $\hat{a}_2$: $|A_{11}|^2 - |A_{12}|^2 = |A_{21}|^2 - |A_{22}|^2 = 1$

$$A_{11}^* A_{21} - A_{12}^* A_{22} = 0$$

That also implies $\hat{a}_1'^+ \hat{a}_1' - \hat{a}_2^{+\prime} \hat{a}_2' = \hat{a}_1^+ \hat{a}_1 - \hat{a}_2^+ \hat{a}_2$

photon difference number is conserved

(photons are created in pairs)

Note that the total number of photons $\hat{a}_1^+ \hat{a}_1' + \hat{a}_2^+ \hat{a}_2$ is not conserved due to amplification

In case of real A_{ij} coefficients, one can present the transformation matrix A as

$$A = \begin{pmatrix} \cosh(\theta/2) & \sinh(\theta/2) \\ \sinh(\theta/2) & \cosh(\theta/2) \end{pmatrix}$$

(if A_{ij} are complex, it can be ~~so~~ included as two separate phase shifters before and after the amplification matrix)

Two quadratures are transformed in a similar way

Note: I will use q and p instead of X_1 and X_2 to avoid confusion with two channel $q \equiv X_1$, $p \equiv X_2$

$$\begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \begin{pmatrix} \cosh\theta/2 & \sinh\theta/2 \\ \sinh\theta/2 & \cosh\theta/2 \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix}$$

$$\begin{pmatrix} \hat{p}_1' \\ \hat{p}_2' \end{pmatrix} = \begin{pmatrix} \cosh\theta/2 & -\sinh\theta/2 \\ -\sinh\theta/2 & \cosh\theta/2 \end{pmatrix} \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix}$$

Using $\begin{pmatrix} \cosh\theta/2 & \sinh\theta/2 \\ \sinh\theta/2 & \cosh\theta/2 \end{pmatrix} = R^{-1} \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} R$

$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ 45° rotation matrix

$$\hat{R}\times \begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \overset{\hat{R}\times}{R^{-1}} \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} R \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix}$$

$$\hat{R}\times \begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{q}_1' \\ \hat{q}_2' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{q}_1' + \hat{q}_2' \\ \hat{q}_1' - \hat{q}_2' \end{pmatrix} = \begin{pmatrix} \hat{q}_+' \\ \hat{q}_-' \end{pmatrix}$$

$$\hat{q}_\pm' = \frac{\hat{q}_1' \pm \hat{q}_2'}{\sqrt{2}}$$ Rotated joint quadrature

Thus, for the joint quadratures

$$\begin{pmatrix} \hat{q}'_+ \\ \hat{q}'_- \end{pmatrix} = \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} \begin{pmatrix} \hat{q}_+ \\ \hat{q}_- \end{pmatrix}$$

or $\hat{q}'_- = e^{-\theta/2}\, \hat{q}_-$

For strong amplification $\theta \to \infty$ $e^{-\theta/2} \to 0$

thus $\hat{q}'_- \to 0$

$\hat{q}_1 - \hat{q}_2 \to 0$

Similarly $\hat{p}_1 + \hat{p}_2 \to 0$

Quadratures are strongly correlated, which is the indication of entanglement. Thus, measurements of the first field allow predict with certainty the measurements for the second

since $\hat{q}_1 = \hat{q}_2$

and $\hat{p}_1 = -\hat{p}_2$