What if we have more than two states?

$\hat{H}_0 : \quad E_n^{(0)}, \; \psi_n^{(0)}$

Time dependent Schrodinger equation:

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \left(\hat{H}_0 + \hat{H}'(t) \right) \psi(t)$$

Now: $\psi(t) = \sum_k c_k(t) \psi_k e^{-iE_k t/\hbar}$

$$\frac{\partial \psi(t)}{\partial t} = \sum_k \dot{c}_k \psi_k e^{-iE_k t/\hbar} + \underbrace{\frac{1}{i\hbar} \sum_k c_k(t) E_k \psi_k e^{-iE_k t/\hbar}}$$

these terms will cancel out with $\hat{H}_0 \psi$ terms in the Schrodinger equation

Remaining terms

$$i\hbar \sum_k \dot{c}_k \psi_k e^{-iE_k t/\hbar} = \sum_k c_k(t) \left(\hat{H}' \psi_k \right) e^{-iE_k t/\hbar}$$

We are going to use the orthonormality of the wave functions again to isolate a single $\dot{c}_n$ coefficient $\langle \psi_n | \psi_k \rangle = \delta_{nk}$

$$i\hbar \dot{c}_n e^{-iE_n t/\hbar} = \sum_k c_k(t) \langle \psi_n | \hat{H}' | \psi_k \rangle e^{-iE_k t/\hbar}$$

$$\dot{c}_n = -\frac{i}{\hbar} \sum_k c_k(t) \langle \psi_n | \hat{H}' | \psi_k \rangle e^{-i(E_k - E_n)t/\hbar}$$

$$\dot{c}_n = -\frac{i}{\hbar} \sum_k c_k(t) H'_{nk}(t) e^{+i\omega_{nk} t} \qquad \omega_{nk} = \frac{E_n - E_k}{\hbar}$$

Exact ~~equa~~ system of equation, describing the state evolution.

Perturbation approximation: each $c_n = \overset{\text{assume } H'=0}{c_n^{(0)}} + \underset{\sim H'}{c_n^{(1)}} + \underset{\sim (H')^2}{c_n^{(2)}} + \dots$

Zero approximation: $\dot{c}_n = 0$ for all n, the initial state decomposition is not affected. $c_n^{(0)}(t) = c_n(0)$

First order: $\dot{c}_n(t) = -\frac{i}{\hbar} \sum_k c_k^{(0)} H'_{nk}(t)\, e^{i\omega_{nk}t}$

$$c_n^{(1)}(t) = -\frac{i}{\hbar} \sum_k c_k(t=0) \int_0^t H'_{nk}(t')\, e^{i\omega_{nk}t'}\, dt'$$

Very often we know ~~what~~ that a system is in one of the unperturbed states ψ_i at $t=0$ ("i" here stands for "initial")

Then $c_n = \delta_{ni}$, and

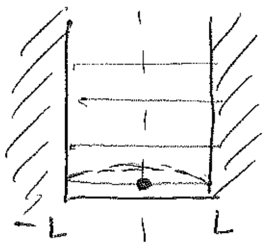
$$c_n^{(1)}(t) = -\frac{i}{\hbar} \int_0^t H'_{ni}(t')\, e^{i\omega_{ni}t'}\, dt' \qquad \text{for any } n \neq i$$

$$c_i^{(1)}(t) = 0$$

This expression is very similar to what we had for a two-level system.

Example: infinite square well with a kick!

Main hamiltonian: $U(x) = \begin{cases} 0 & -L<x<L \\ \infty & |x|>L \end{cases}$



No perturbation at $t<0$, the system is in the ground state $n=1$.

At $t=0$ we turn on the perturbation

$$U(x) = \alpha\, \delta(x)\, e^{-t/t_0}$$

(i.e. the perturbation is the strongest at $t=0$, and then slowly vanishes).

What are the probability of finding the particle in various excited states? as time goes on?

Initial state $\psi_i = \psi_1 = \frac{1}{\sqrt{L}} \cos\frac{\pi x}{2L}$ $\quad E_1 = \frac{\hbar^2 \pi^2}{8mL^2}$

For any excited state $n>1$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ni}(t')\, e^{i\omega_{ni}t'}\, dt' = -\frac{i}{\hbar}\int_0^t \langle\psi_n|\alpha\delta(x)|\psi_1\rangle e^{-t'/t_0 + i\omega_{ni}t'}\, dt'$$

$$\langle\psi_n|\alpha\delta(x)|\psi_1\rangle = \frac{\alpha}{\sqrt{L}}\int_{-L}^{L}\psi_n(x)\,\delta(x)\cos\frac{\pi x}{2L}\, dx = 0 \text{ for } \cancel{\text{odd}}\ \text{even } n$$

no transitions to those states

for ~~even~~ odd n:

$$\langle\psi_n|\alpha\delta(x)|\psi_1\rangle = \frac{\alpha}{L}\int_{-L}^{L}\delta(x)\cos\frac{\pi n x}{2L}\cos\frac{\pi x}{2L}\, dx = \frac{\alpha}{L}\cos\frac{\pi n}{2}\cos\frac{\pi}{2} = (-1)^{\frac{n-1}{2}}\frac{\alpha}{L}$$

For the odd states

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\frac{\alpha}{L}(-1)^{\frac{n-1}{2}}\int_0^t e^{-t'/t_0 + i\omega_{ni}t'}\, dt' = -\frac{i}{\hbar}\frac{\alpha}{L}(-1)^{\frac{n-1}{2}} \times$$

$$\times \int_0^t e^{t'(-1/t_0 + i\omega_{ni})}\, dt' = -\frac{i}{\hbar}\frac{\alpha}{L}(-1)^{\frac{n-1}{2}} \left.\frac{e^{t'(-1/t_0 + i\omega_{ni})}}{(-1/t_0 + i\omega_{ni})}\right|_0^t$$

$$c_n^{(1)} = -\frac{i}{\hbar}\frac{d}{L}(-1)^{\frac{n-1}{2}} \frac{\left(e^{t(-1/t_0+i\omega_{ni})}-1\right)}{(-1/t_0+i\omega_{ni})}$$

Probability $$p_n(t) = |c_n^{(1)}(t)|^2 = \frac{d^2}{\hbar^2 L^2}\frac{\left|e^{t(-1/t_0+i\omega_{ni})}-1\right|^2}{\left|-1/t_0+i\omega_{ni}\right|^2}$$

$$(1-e^{-t/t_0+i\omega_{ni}t})(1-e^{-t/t_0-i\omega_{ni}t}) = 1+e^{-2t/t_0}-2e^{-t/t_0}\cos\omega_{ni}t$$

$$p_n(t) = \frac{d^2 t_0^2}{\hbar^2 L^2}\frac{\left(1+e^{-2t/t_0}-2e^{-t/t_0}\cos\omega_{ni}t\right)}{1+t_0^2\omega_{ni}^2}$$

For very short times $t \ll t_0$ (at the very beginning)

$$e^{-t/t_0} \approx 1(-t/t_0)$$

$$1+e^{-2t/t_0}-2e^{-t/t_0}\cos\omega_{ni}t \approx 2-2\cos\omega_{ni}t = 4\sin^2\frac{\omega_{ni}t}{2}$$

$$p_n(t) \approx \frac{4d^2t_0^2}{\hbar^2 L^2}\frac{\sin^2\frac{\omega_{ni}t}{2}}{1+t_0^2\omega_{ni}^2}$$ ~~the probabilities are growing~~

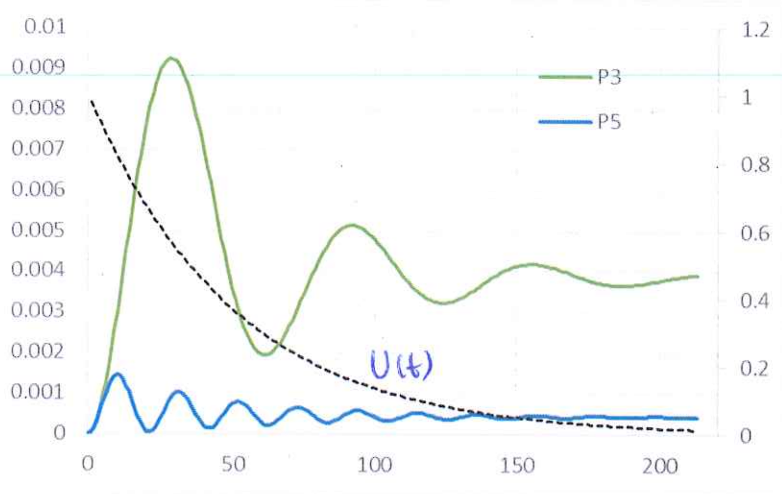
if $t_0\omega_{ni} \ll 1$ $\quad \sin\frac{\omega_{ni}t}{2} \approx \frac{\omega_{ni}t}{2}$

$$p_n(t) \approx \frac{d^2t_0^2}{\hbar^2L^2}\omega_{ni}^2 t^2$$ ← all probabilities are growing quadratically in time at the first moment

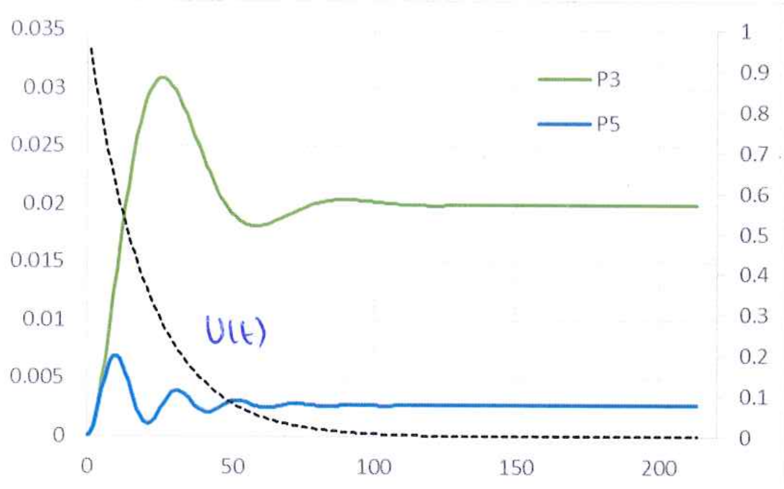
For a very long time $t \gg t_0$ (after all the action)

$e^{-t/t_0} \to 0$ $\qquad p_n(t) = \frac{d^2t_0^2}{\hbar^2L^2}\frac{1}{1+t_0^2\omega_{ni}^2}$ no time dependence

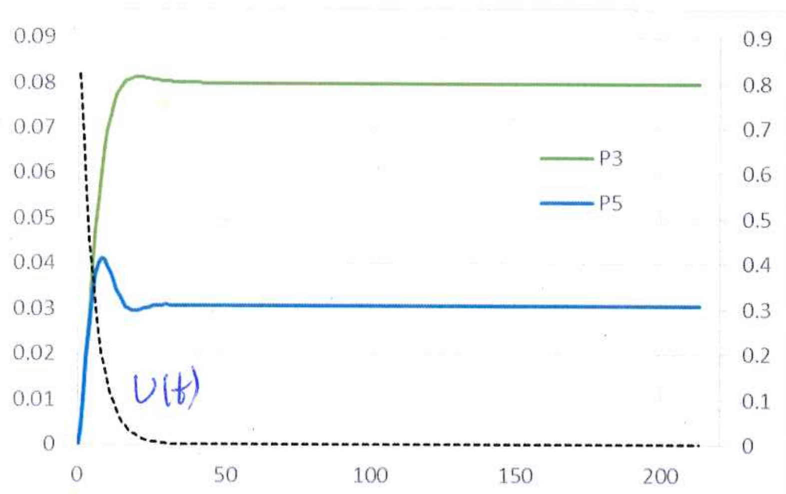
$$\boxed{\begin{aligned}\omega_{13} &= \frac{1}{10}\\ \omega_{15} &= \frac{3}{10}\end{aligned}}$$



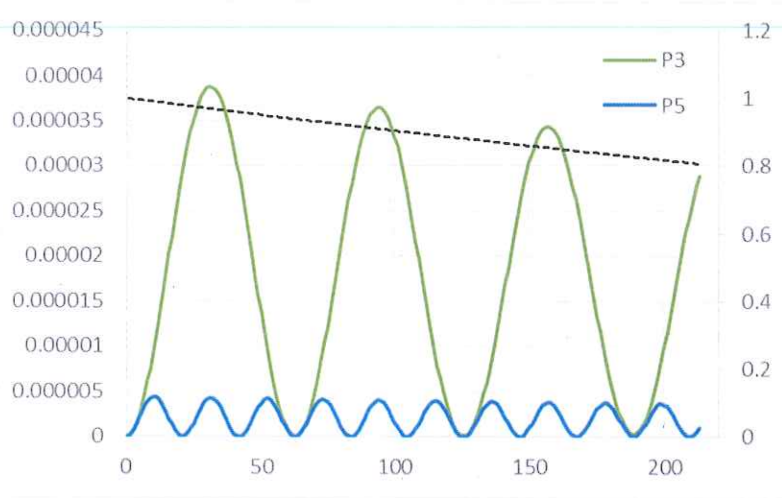
$t_0 = 50$



$t_0 = 20$



$t_0 = 5$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0.00003
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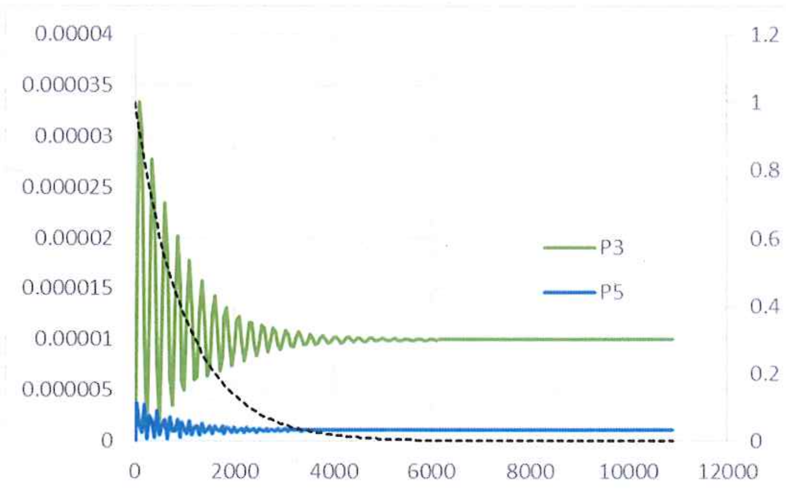
0.00002
0.000015
0.00001
0.000005
0
1.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
P3
P5

$t_0 = 1000$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0.4
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0
0
2000
4000
6000
8000
10000
12000
P3
P5